

LIE SYMMETRY REDUCTION FOR HUXLEY EQUATION

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Abstract

In this paper, Huxley equation which is in the form of second-order nonlinear partial differential equation is transformed to an ordinary differential equation by using Lie symmetry analysis. The procedure of transformation is based on reducing the number of independent variables in partial differential equation by one by using the infinitesimal generator. After calculating invariants by solving the characteristic system and designate one of the invariant as a function of the other, Huxley equation can be written in the form of the ordinary differential equation.

Introduction

The real-life phenomena around us can be described by many fields such as Physics, Chemistry, Biology and others. Since phenomena can be formed into the nonlinear partial differential equations, the solutions of these problems will be interested. Exact solutions for these problems are usually difficult to find, however many analytical techniques are still used to construct these solutions for the proper understanding in phenomena [6].

One of the mathematical disciplines which are used to find the exact solutions of differential equations is Lie group theory. In sometimes, symmetry analysis is another name for calling this method. This method can be used to transform the partial differential equation into ordinary differential equation. Clarkson and Mansfield [7] had studied some of nonlinear heat equations. Symmetry reductions of the partial differential equations and exact solutions

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can be found by using symmetry analysis. In many researches [5, 6, 7 and 10], symmetry analysis are applied to the nonlinear equation. For example, the Fitzhugh-Nagoma nerve conduction equation is one of special cases in generalized nonlinear Burgers-Huxley equation which has been applied by the Lie point symmetry generators [10].

In biology, the generalized nonlinear Burgers-Huxley equation is used to describe the interaction between reaction mechanisms, convection effects and diffusion transports. Two known cases of these evolution equations such as Fitzhugh-Nagoma equation and Huxley equation are reduced from the generalized nonlinear Burgers-Huxley equation [4]. Several methods have been developed to solve Burgers-Huxley equation; for example, Differential Transform Method (DTM) [4], Homotopy Analysis Method (HAM) [1, 2], First Integral Method [9], He's Variational Iteration Method [3], He's Homotopy Perturbation Method [8], Symmetry Analysis [5, 6, 7, and 10], Tanh method [11] and others.

In this work, Lie group theory is applied to transform the Huxley equation so as to provide us an ordinary differential equation that may be easier to solve. First of all, the infinitesimal generators are the tool for reducing the number of independent variables in partial differential equation by one, so the equation will have fewer independent variables. After that the characteristic system will be used to calculate invariants and then we will designate one of the invariant as a function of the other. Finally, Huxley equation can be transformed to the ordinary differential equation by substituting the dependent variables.

Huxley equation

The nerve propagation in biology can be described by Huxley equation which is one of special cases of the generalized nonlinear Burgers-Huxley equation [4].

$$u_t - u_{xx} = u^{2\delta+1} + (1 + \gamma)u^{\delta+1} - \gamma u \quad (1)$$

where $0 \leq x \leq 1, t \geq 0$ and constants $\gamma \neq 0, \delta \geq 2$.

Lie group theory and the transformation In this section, we will apply Lie symmetry analysis to Huxley equation.

First of all, an infinitesimal generator for Huxley equation is written in the following form

$$V = \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u} \quad (2)$$

The coefficient functions $\xi(x, t, u), \eta(x, t, u)$ and $\phi(x, t, u)$ will be determined because this leading tool will be used for generating the symmetry group of equation (1). Since Huxley equation is the second order partial differential equation, the second prolongation of the generator (2) will be determined

$$pr^{(2)}V = V + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} \quad (3)$$

where

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, \quad (4)$$

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{xxt} \quad (5)$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \quad (6)$$

The total derivatives with respect to x, t will be used to determine the coefficients of $pr^{(2)}V$, respectively.

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + u_{xxt} \frac{\partial}{\partial u_{xt}} + u_{xtt} \frac{\partial}{\partial u_{tt}} + \dots \quad (7)$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xxt} \frac{\partial}{\partial u_{xx}} + u_{xtt} \frac{\partial}{\partial u_{xt}} + u_{ttt} \frac{\partial}{\partial u_{tt}} + \dots \quad (8)$$

$$D_{xx} = D_x(D_x), D_{xt} = D_x(D_t), D_{tt} = D_t(D_t) \quad (9)$$

The system for determining the coefficient functions $\xi(x, t, u), \eta(x, t, u)$ and $\phi(x, t, u)$ are calculated by the standard symmetry group, we have

$$\xi = k_1 \eta = k_2; \phi = 0 \quad (10)$$

Therefore, the infinitesimal generators admitted by Huxley equation have the form

$$V = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial t} \quad (11)$$

where k_1 and k_2 are arbitrary constants.

The Lie algebra corresponding to Huxley equation is spanned by the second linearly independent generators

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t} \quad (12)$$

In this section, Huxley equation will be transformed to ordinary differential equation by using the infinitesimal generators obtained in (9).

The dependent variable u will be expressed as a function of invariant $I_1 = u$ and another invariant for generator V is found from the characteristic system

$$\frac{dx}{1} = \frac{dt}{1} \quad (13)$$

By integrating (13), we get $I_2 = x - t$.

Let $\nu = u$, $y = x - t$ and then the derivatives of u are calculated.

$$u_t = \left(\frac{d\nu}{dy}\right)\left(\frac{dy}{dt}\right) = (\nu')(-1) = -\nu' \quad (14)$$

$$u_x = \left(\frac{d\nu'}{dy}\right)\left(\frac{\partial y}{\partial x}\right) = (\nu')(1) = \nu' \quad (15)$$

$$u_{xx} = \left(\frac{d\nu''}{dy}\right)\left(\frac{\partial y}{\partial x}\right) = (\nu'')(1) = \nu'' \quad (16)$$

After substituting these derivatives into (1), Huxley equation is transformed to an ordinary differential equation as follows.

$$\nu'' = \nu^{2\delta+1} - (1 + \gamma)\nu^{\delta+1} + \gamma\nu - \nu'. \quad (17)$$

Concluding remarks

Lie symmetry analysis is applied to the Huxley equation for reducing the number of the independent variable. After reducing the number of the independent variable by one by using the infinitesimal generators, the second-order nonlinear partial differential equation can be transformed to an ordinary differential equation.

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