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ON A RESPONSE OF THE STOCHASTIC RAYLEIGH SYSTEM

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Abstract

The paper shows that the response of the random Rayleigh system, which is under harmonic and random excitations, can be found by a system of algebraic equations. The analytical approach is based on the stochastic averaging method and equivalent linearization method in Cartesian coordinates so that the Fokker-Planck equation associated with the linear equations obtained can be solved exactly by the technique of auxiliary function. The harmonic excitation frequency is taken to be in the neighborhood of the system natural frequency. The mean-square responses obtained by the proposed approach are compared with those obtained by Monte Carlo simulation method.

1 Introduction

Systems under harmonic excitation and (or) random excitation have received a flurry of research effort in the past few decades. Under purely harmonic excitation, it is common to use the technique of averaging method. Over years, the stochastic averaging method has proved to be a powerful approximate technique for the prediction of response of weakly nonlinear vibrations subjected to dampings and random excitations [1-6]. Comprehensive reviews attesting the success of the stochastic averaging method in random vibration have been done by Roberts and Spanos [7]. The advantage of this method is that the

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equations of motion of a system are much simplified and the dimensions of the response coordinates are often reduced, and that the averaged response is a diffusive Markov process and the method of Fokker-Planck (FP) equation, whose exact solution is available just for some special cases [8-10], can be applied. To solve FP equation, several approximate and numerical techniques have been developed [11-17]. Under purely random excitation, along with the stochastic averaging method, the stochastic equivalent linearization method is another popular approach to the approximate analysis. The method consists of optimally approximating the non- linearities in the given system by linear models so that the solution of the resulting equivalent system is available. The original version of this method was first proposed by Caughey [18,19] and has been developed up to recent years by many authors [20-25]. Engineering systems, however, are often subjected to combined harmonic and random excitations, and their exact solutions are known only for a number of special cases. Therefore, the combination of various methods plays an important role in order to find responses of such systems. Some methods (or techniques), such as the combination of the averaging method and Fokker-Planck equation [8,9], the averaging method and technique of auxiliary function for FP equation [4,8,26,27], the method of multiple scales and second-order closure method [28], the averaging with equivalent nonlinearization technique [29], averaging method, FP equation and the path integration [30-32], the method of harmonic balance and the method of stochastic averaging [33], the averaging method and linearization method [34], have been used for the analyses. To our knowledge, the stochastic Rayleigh system under combined harmonic and random excitations hasnt been investigated so far. In this paper, the approximate technique, proposed by Anh et al [35], is employed for the Rayleigh with weak nonlinearity and weak excitations. The key concept of the approach is that the stochastic averaging the original equation is carried out in Cartesian coordinates and that the technique of auxiliary function for FP equation. By using the conventional equivalent linearization method, the nonlinear averaged equations can be replaced by linear ones whose solutions can be found exactly by the technique of auxiliary function.

2 The stochastic Rayleigh system

Let's consider a Rayleigh oscillator subject to harmonic and random excitations. The equation of motion of the system is of the form

$$\ddot{x} + \omega^2 x = \varepsilon\left(\dot{x} - \gamma\dot{x}^3\right) + \varepsilon Q\cos\nu t + \sqrt{\varepsilon}\sigma\xi\left(t\right) \tag{1}$$

where ω, ν, γ, Q, σ are constants, ε is a small positive parameter, $\xi\left(t\right)$ is a Gaussian white noise process of unit intensity with the correlation function $R_\xi\left(\tau\right) = \langle\xi\left(t\right)\xi\left(t+\tau\right)\rangle = \delta\left(\tau\right)$, where $\delta\left(\tau\right)$ is the Dirac delta function, and notation $\langle . \rangle$ denotes the mathematical expectation operator. In primary resonant frequency region, parameters ω and ν have the relation

$$\omega^2 - \nu^2 = \varepsilon\Delta, \tag{2}$$

where Δ is a detuning parameter.

2.1 The system of algebraic equations for the approximate stationary probability density function (PDF) of the system

We seek the solution of Eq. (1) in the form of

$$x = a_1 \cos\varphi + a_2 \sin\varphi, \ \ \dot{x} = -a_1\nu \sin\varphi + a_2\nu\cos\varphi, \ \ \varphi = \nu t, \tag{3}$$

where a_1 and a_2 are slowly varying random processes. Applying Ito rule and using stochastic averaging method yield

$$\begin{aligned} \dot{a}_1 &= \varepsilon\langle K_1\rangle_t + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}} B_1(t), \\ \dot{a}_2 &= \varepsilon\langle K_2\rangle_t + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}} B_2(t), \end{aligned} \tag{4}$$

where $\langle . \rangle_t$ is the averaging operator with respect to time t, $B_1(t)$ and $B_2(t)$ are independent Gaussian white noises, and, with $f = -\Delta x + \dot{x} - \gamma\dot{x}^3 + Q\cos\nu t$,

$$\begin{aligned} \langle K_1\rangle_t &= -\frac{1}{\nu}\langle f\sin\varphi\rangle_t = \frac{1}{2}a_1 + \frac{\Delta}{2\nu}a_2 - \frac{3}{8}\gamma\nu^2\left(a_1^3 + a_1a_2^2\right), \\ \langle K_2\rangle_t &= \frac{1}{\nu}\langle f\cos\varphi\rangle_t = -\frac{\Delta}{2\nu}a_1 + \frac{1}{2}a_2 + \frac{Q}{2\nu} - \frac{3}{8}\gamma\nu^2\left(a_1^2a_2 + a_2^3\right). \end{aligned} \tag{5}$$

The FP equation, written for the stationary probability density function (PDF) $p(a_1, a_2)$ associated with the system (4), has the form

$$\frac{\partial}{\partial a_1}\left(\langle K_1\rangle_t p\right) + \frac{\partial}{\partial a_2}\left(\langle K_2\rangle_t p\right) = \frac{\sigma^2}{4\nu^2}\left[\frac{\partial^2 p}{\partial a_1^2} + \frac{\partial^2 p}{\partial a_2^2}\right]. \tag{6}$$

So far, an exact solution of FP equation (6) is only available for a very limited number of problems; nevertheless, if functions $\langle K_1\rangle_t$, $\langle K_2\rangle_t$ are linear functions then Eq. (6) can be solved exactly by the technique of auxiliary function [10]. Further, it is seen that the transformation (3) makes the drift coefficients $\langle K_1\rangle_t$, $\langle K_2\rangle_t$ given in (5) be polynomials in a_1 and a_2 which give an advantageous context to apply the equivalent linearization method. Thus, the method of linearization is employed here. Following this method, the functions $\langle K_1\rangle_t$, $\langle K_2\rangle_t$ in (6) are replaced by linear functions H_i, $i = 1, 2$ given by

$$\begin{aligned} H_1(a_1, a_2) &= \alpha_1 a_1 + \beta_1 a_2 + \lambda_1, \\ H_2(a_1, a_2) &= \alpha_2 a_1 + \beta_2 a_2 + \lambda_2. \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\alpha_1 = \frac{1}{2} + \eta_{11}, \quad \beta_1 = \frac{\Delta}{2\nu} + \eta_{12}, \ \lambda_1 = \eta_{13}, \\ &\alpha_2 = -\frac{\Delta}{2\nu} + \eta_{21}, \beta_2 = \frac{1}{2} + \eta_{22}, \ \lambda_2 = \frac{Q}{2\nu} + \eta_{23}. \end{aligned} \tag{8}$$

There are some criteria for determining the coefficients α_i, β_i, λ_i. The most extensively used criterion is the mean square error criterion which requires that the mean square of errors be minimum [18]. Errors between the nonlinear functions $\langle K_i \rangle_t$ and the linear functions H_i, $i = 1, 2$ are

$$e_i = \langle K_i \rangle_t - (\alpha_i a_1 + \beta_i a_2 + \lambda_i), \ i = 1, 2. \tag{9}$$

So, the mean square error criterion leads to

$$\langle e_i^2 \rangle \to \min_{\alpha_i, \beta_i, \lambda_i}, \ i = 1, 2. \tag{10}$$

From

$$\frac{\partial}{\partial \alpha_i} \langle e_i^2 \rangle = 0, \ \frac{\partial}{\partial \beta_i} \langle e_i^2 \rangle = 0, \ \frac{\partial}{\partial \lambda_i} \langle e_i^2 \rangle = 0, \ \ i = 1, 2, \tag{11}$$

it follows that

$$\begin{aligned} &\langle a_1 \langle K_1 \rangle_t \rangle - \langle a_1^2 \rangle \alpha_1 - \langle a_1 a_2 \rangle \beta_1 - \langle a_1 \rangle \lambda_1 = 0, \\ &\langle a_2 \langle K_1 \rangle_t \rangle - \langle a_1 a_2 \rangle \alpha_1 - \langle a_2^2 \rangle \beta_1 - \langle a_2 \rangle \lambda_1 = 0, \\ &\langle \langle K_1 \rangle_t \rangle - \langle a_1 \rangle \alpha_1 - \langle a_2 \rangle \beta_1 - \lambda_1 = 0, \\ &\langle a_1 \langle K_2 \rangle_t \rangle - \langle a_1^2 \rangle \alpha_2 - \langle a_1 a_2 \rangle \beta_2 - \langle a_1 \rangle \lambda_2 = 0, \\ &\langle a_2 \langle K_2 \rangle_t \rangle - \langle a_1 a_2 \rangle \alpha_2 - \langle a_2^2 \rangle \beta_2 - \langle a_2 \rangle \lambda_2 = 0, \\ &\langle \langle K_2 \rangle_t \rangle - \langle a_1 \rangle \alpha_2 - \langle a_2 \rangle \beta_2 - \lambda_2 = 0. \end{aligned} \tag{12}$$

The relations (12), solved with respect to α_i, β_i, λ_i, then, from (8) reads

$$\begin{aligned} &\eta_{11} = -\frac{3\gamma\nu^2}{8} \left(3\sigma_{a_1}^2 + 3\langle a_1 \rangle^2 + \sigma_{a_2}^2 + \langle a_2 \rangle^2 \right), \ \eta_{12} = -\frac{3\gamma\nu^2}{4} \left(\langle a_1 \rangle \langle a_2 \rangle + k_{a_1 a_2} \right), \\ &\eta_{13} = \frac{3\gamma\nu^2}{4} \left(\langle a_1 \rangle^2 + \langle a_2 \rangle^2 \right) \langle a_1 \rangle, \qquad \eta_{21} = -\frac{3\gamma\nu^2}{4} \left(\langle a_1 \rangle \langle a_2 \rangle + k_{a_1 a_2} \right), \\ &\eta_{22} = -\frac{3\gamma\nu^2}{8} \left(\sigma_{a_1}^2 + \langle a_1 \rangle^2 + 3\sigma_{a_2}^2 + 3\langle a_2 \rangle^2 \right), \ \eta_{23} = \frac{3\gamma\nu^2}{4} \left(\langle a_1 \rangle^2 + \langle a_2 \rangle^2 \right) \langle a_2 \rangle. \end{aligned} \tag{13}$$

Furthermore, if the system (4) is linear and under Gaussian process excitation, one gets that a_1 and a_2 are jointly Gaussian. Thus, all higher moments of a_1 and a_2 in (12) and (13) can be expressed in terms of the first and second moments of a_1 and a_2 by below properties of a Gaussian random vector $\vec{X} = (a_1, a_2)$

$$\begin{aligned} &\langle a_i^{n+1} \rangle = \langle a_i \rangle \langle a_i^n \rangle + n\sigma_{a_i}^2 \langle a_i^{n-1} \rangle, \\ &\langle a_i a_1^{n_1} a_2^{n_2} \rangle = \langle a_i \rangle \langle a_1^{n_1} a_2^{n_2} \rangle + n_1 k_{a_i a_1} \langle a_1^{n_1 - 1} a_2^{n_2} \rangle + n_2 k_{a_i a_2} \langle a_1^{n_1} a_2^{n_2 - 1} \rangle, \ i = 1, 2. \end{aligned} \tag{14}$$

Here $\sigma_{a_i}^2$ is a variance of a_i, $k_{a_1a_2}$ denotes a covariance of a_1 and a_2, and n,n_1 and $n_2 = 0, 1, 2, ...$ Thus, the relation (12) results in six algebraic equations for eleven unknowns: α_i, β_i, $\lambda_i (i = 1, 2)$, $\langle a_1 \rangle$, $\langle a_2 \rangle$, $\sigma_{a_1}^2$, $\sigma_{a_2}^2$, $k_{a_1a_2}$. To close the system (12), more relations of the unknowns are needed. It is noted that the FP equation below written for the stationary PDF $p(a_1, a_2)$ associated with the system (4) has the following form

$$\frac{\partial}{\partial a_1}(H_1 p) + \frac{\partial}{\partial a_2}(H_2 p) = \frac{\sigma^2}{4\nu^2}\left[\frac{\partial^2 p}{\partial a_1^2} + \frac{\partial^2 p}{\partial a_2^2}\right] \tag{15}$$

In order to integrate Eq.(6) in which the functions $\langle K_1 \rangle_t$, $\langle K_2 \rangle_t$ are replaced by linear functions H_i, $i = 1, 2 (7)$, we employ the technique of auxiliary function [10] with the auxiliary function $u(a_1, a_2) = u_0 = const$ (in case of $\alpha_1 + \beta_2 \neq 0$) as follows

$$u_0 = \frac{\sigma^2}{4\nu^2}\frac{(\alpha_2 - \beta_1)}{\alpha_1 + \beta_2} \tag{16}$$

Then the stationary PDF $p(a_1, a_2)$ can be found in the form of

$$p(a_1, a_2) = C \exp\left\{-\tau_1 a_1^2 - \tau_2 a_2^2 + \tau_3 a_1 a_2 + \tau_4 a_1 + \tau_5 a_2\right\} \tag{17}$$

where

$$\begin{aligned}
\tau_1 &= -\frac{2\nu^2(\alpha_1 + \beta_2)}{\sigma^2\left[(\alpha_2 - \beta_1)^2 + (\alpha_1 + \beta_2)^2\right]}\left[\alpha_1(\alpha_1 + \beta_2) + \alpha_2(\alpha_2 - \beta_1)\right], \\
\tau_2 &= -\frac{2\nu^2(\alpha_1 + \beta_2)}{\sigma^2\left[(\alpha_2 - \beta_1)^2 + (\alpha_1 + \beta_2)^2\right]}\left[(\alpha_1 + \beta_2)\beta_2 + (-\alpha_2 + \beta_1)\beta_1\right], \\
\tau_3 &= \frac{4\nu^2(\alpha_1 + \beta_2)}{\sigma^2\left[(\alpha_2 - \beta_1)^2 + (\alpha_1 + \beta_2)^2\right]}(\alpha_1\beta_1 + \alpha_2\beta_2), \\
\tau_4 &= \frac{4\nu^2(\alpha_1 + \beta_2)}{\sigma^2\left[(\alpha_2 - \beta_1)^2 + (\alpha_1 + \beta_2)^2\right]}\left[\lambda_1(\alpha_1 + \beta_2) + \lambda_2(\alpha_2 - \beta_1)\right], \\
\tau_5 &= \frac{4\nu^2(\alpha_1 + \beta_2)}{\sigma^2\left[(\alpha_2 - \beta_1)^2 + (\alpha_1 + \beta_2)^2\right]}\left[\lambda_1(-\alpha_2 + \beta_1) + \lambda_2(\alpha_1 + \beta_2)\right].
\end{aligned} \tag{18}$$

Here, because Eq. (15) is associated with a linear system under Gaussian white noise, the coefficients τ_1 and τ_2 are positive so that the PDF $p(a_1, a_2)$ (17) has a finite integral. It is noted that, from the stationary PDF (17), the moments

$\langle a_1\rangle$, $\langle a_2\rangle$, $\sigma_{a_1}^2$, $\sigma_{a_2}^2$, $k_{a_1a_2}$ can be derived in terms of τ_i, $i=\overline{1,5}$ as

$$\begin{aligned}\langle a_1\rangle &= \frac{2\tau_2\tau_4+\tau_3\tau_5}{4\tau_1\tau_2-\tau_3^2},\\ \langle a_2\rangle &= \frac{2\tau_1\tau_5+\tau_3\tau_4}{4\tau_1\tau_2-\tau_3^2},\\ \sigma_{a_1}^2 &= \frac{2\tau_2}{4\tau_1\tau_2-\tau_3^2},\\ \sigma_{a_2}^2 &= \frac{2\tau_1}{4\tau_1\tau_2-\tau_3^2},\\ k_{a_1a_2} &= \frac{\tau_3}{4\tau_1\tau_2-\tau_3^2}\end{aligned} \tag{19}$$

Thus, from the stationary PDF (17), the moments $\langle a_1\rangle$, $\langle a_2\rangle$, $\sigma_{a_1}^2$, $\sigma_{a_2}^2$, $k_{a_1a_2}$ can be derived in terms of α_i, β_i, λ_i by the relations (18) and (19). And then, the relations (12), (18) and (19) give us a closed system of eleven equations for eleven unknowns α_i, β_i, $\lambda_i(i=1,2)$, $\langle a_1\rangle$, $\langle a_2\rangle$, $\sigma_{a_1}^2$, $\sigma_{a_2}^2$, $k_{a_1a_2}$. After being found by solving the system (12), (18) and (19), the values of the linearization coefficients α_i, β_i, λ_i are substituted into (17) and (18) to obtain the approximate stationary PDF in a_1 and a_2 of Eq. (1).

2.2 The expression of the Rayleigh's response

Taking mathematical expectation both sides of Eq. (3) gives

$$\langle x\,(t)\rangle = \langle a_1\rangle\cos\nu t + \langle a_2\rangle\sin\nu t. \tag{20}$$

Moreover, by squaring both sides of the first equation in (3) and then taking mathematical expectation, one obtains

$$\left\langle x^2\,(t)\right\rangle = \left\langle a_1^2\right\rangle\cos^2\nu t + \left\langle a_2^2\right\rangle\sin^2\nu t + \langle a_1a_2\rangle\sin 2\nu t. \tag{21}$$

It is seen from Eq. (20) and Eq. (21) that expectation of $x\,(t)$ and $x\,(t)$ are periodic in time t. From the expression of PDF (17) and the translation (3), the joint PDF of x and $\dot{x}$ can be written as

$$\begin{aligned}\bar{p}(x,\dot{x},t) =& \frac{C}{\nu}\exp\left\{-\tau_1\left(x\cos\nu t-\frac{\dot{x}}{\nu}\sin\nu t\right)^2-\tau_2\left(x\sin\nu t+\frac{\dot{x}}{\nu}\cos\nu t\right)^2+\right.\\ &+\tau_3\left(x\cos\nu t-\frac{\dot{x}}{\nu}\sin\nu t\right)\left(x\sin\nu t+\frac{\dot{x}}{\nu}\cos\nu t\right)+\\ &\left.\tau_4\left(x\cos\nu t-\frac{\dot{x}}{\nu}\sin\nu t\right)+\tau_5\left(x\sin\nu t+\frac{\dot{x}}{\nu}\cos\nu t\right)\right\}.\end{aligned} \tag{22}$$

From Eq. (22), one gets the marginal PDF of x as

$$\bar{p}(x,t) = \int\limits_{-\infty}^{\infty} \bar{p}(x,\dot{x},t)\, d\dot{x} \tag{23}$$

It is seen from (17), (22) and (23) that the joint PDF of x and $\dot{x}$ and the marginal PDF of x depend on time t, although two variables a_1 and a_2 are described in a stationary joint PDF. Then taking time-averaging Eq. (21) yields

$$\begin{aligned} \left\langle \left\langle x^2(t) \right\rangle \right\rangle_t &= \frac{1}{2\pi} \int\limits_0^{2\pi} \left\langle x^2(t) \right\rangle d(\nu t) \\ &= \frac{1}{2}\left(\left\langle a_1^2 \right\rangle + \left\langle a_2^2 \right\rangle \right) \\ &= \frac{1}{2}\left(\left\langle a_1 \right\rangle^2 + \sigma_{a_1}^2 + \left\langle a_2 \right\rangle^2 + \sigma_{a_2}^2 \right). \end{aligned} \tag{24}$$

Substituting (19) into (24) and reducing the obtained result yield the time-averaging of mean square response to be

$$\left\langle \left\langle x^2(t) \right\rangle \right\rangle_t = \frac{(2\tau_2\tau_4 + \tau_3\tau_5)^2 + (2\tau_1\tau_5 + \tau_3\tau_4)^2}{2(4\tau_1\tau_2 - \tau_3^2)^2} + \frac{\tau_1 + \tau_2}{4\tau_1\tau_2 - \tau_3^2}, \tag{25}$$

where $\tau_i, i = \overline{1,5}$ are given by (18). It is noted from (25) that the approximate time-averaging value of mean square response of the oscillator is calculated algebraically. In Table 1, time-averaging values of mean-square response of the system is performed for computation with various values of the parameterσ^2. In order to check the accuracy of the present technique, the various values of the response of the equation considered $\left\langle x^2 \right\rangle_{present}$ obtained by the proposed technique are compared to the numerical simulation results versus the particular parameter. The numerical simulation of the mean square response, denoted by $\left\langle x^2 \right\rangle_{sim}$, is obtained by 10,000-realization Monte Carlo simulation. The system parameters are chosen to be $\omega = 1$, $Q = 5$, $\gamma = 1$, $\sigma^2 = 0.1$, $\varepsilon = 0.1$, $\nu = 1.01$. It is seen from Table 1 that the proposed technique gives a good prediction. With the same values for system parameters, Figure 1 gives us a plot of the marginal PDF $\bar{p}(x,t)$ at $t = 298$.

Table 1. The error between the simulation result and approximate values of the time-averaging of mean square response $\left\langle x^2(t) \right\rangle$ versus the parameter σ^2 $\left(\omega = 1, Q = 5, \gamma = 1, \ \sigma^2 = 0.1, \varepsilon = 0.1, \nu = 1.01 \right)$.

σ^2	$\langle x^2 \rangle_{sim}$	$\langle x^2 \rangle_{present}$	Err (%)
0.1	2.1662	2.1923	1.2
1	2.1714	2.1846	0.61
2	2.1789	2.1731	0.27
3	2.1899	2.1600	1.36
4	2.2017	2.1461	2.53
5	2.2185	2.1322	3.89

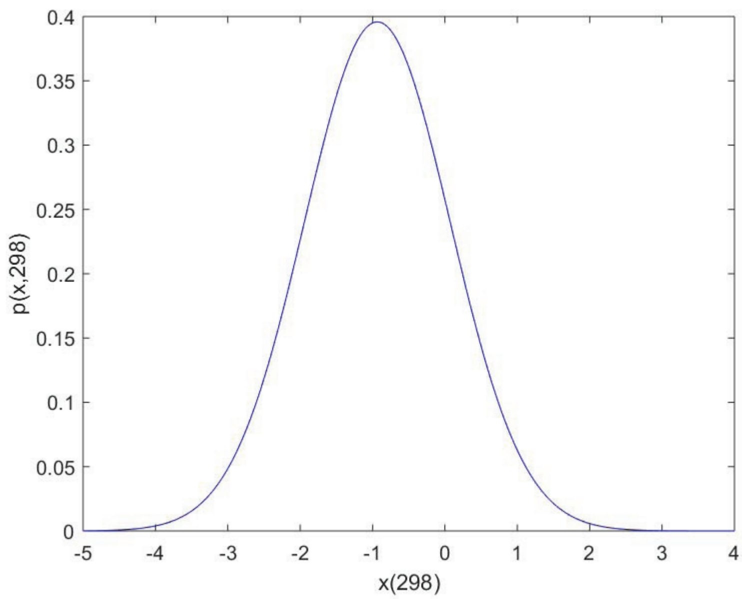

Figure 1: The PDF of x at $t = 298$ with $\omega = 1, Q = 5, \gamma = 1, \sigma^2 = 0.1, \varepsilon = 0.1, nu = 1.01$.

3 Summary and conclusions

The paper shows that Rayleigh's responses can be found from the system of algebraic equations. The main idea of the technique is summarized as follows. First, the stochastic averaging of the equation is carried out in Cartesian coordinates (a_1, a_2) by the transformation (3). The drift coefficients of the averaged equations in the system (6) then are polynomial forms in a_1 and a_2 which give an advantageous context to apply stochastic equivalent linearization method. The linearization coefficients are determined by a closed system. The FP equation associated with the equivalent linearized system can be solved exactly by the technique of auxiliary function. The proposed technique has been applied to Rayleigh and Duffing oscillators under periodic and random excitations. It is found from these applications that the approximate stationary solutions and simulation results agree quite well. The procedure of this technique can be performed in five steps:

- Step 1: Stochastic averaging method in Cartesian coordinates.
- Step 2: Equivalent linearization method to nonlinear FP.
- Step 3: Get the solution of the linear FP by technique of auxiliary function.
- Step 4: Find the approximate stationary PDF of the system from the system of algebraic equations.
- Step 5: Get the approximate responses values.

When using this technique to study an arbitrary nonlinear system, a question about the accuracy of the technique may arise. However, this can be solved if advanced methods of averaging and equivalent linearization are used. This could be explored in future studies.

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References

[1] N.M. Krylov, N.N. Bogoliubov, Introduction to nonlinear mechanics. Ukraine: Academy of Sciences; 1937.

[2] N.N. Bogoliubov, Yu.A. Mitropolskii, Asymptotic methods in the theory of nonlinear oscillations. Moscow: Nauka; 1963.

[3] Y.A. Mitropolskii, Averaging method in non-linear mechanics, Int. J. Non-linear Mechanics. Pergamoa Press Ltd. 2 (1967) 69-96.

[4] Mitropolskii YA, Dao NV, Anh ND. Nonlinear oscillations in systems of arbitrary order. Kiev: Naukova- Dumka (in Russian) 1992.

[5] R.L. Stratonovich, Topics in the Theory of Random Noise, Vol. II, New York: Gordon and Breach, 1967.

[6] R.Z. Khasminskii, A limit theorem for the solutions of differential equations with random right-hand sides, Theory Probab. Applic. 11 (1966) 390-405.

[7] J.B. Roberts, P.D. Spanos, Stochastic averaging: An approximate method of solving random vibration problems, Int. J. Non-linear mechanics 21(2) (1986) 111-134.

[8] M.F. Dimentberg, Response of a non-linearly damped oscillator to combined periodic parametric and random external excitation, Int. J. Non-linear Mechanics 11 (1976) 83-87.

[9] M.F. Dimentberg, An exact solution to a certain non-linear random vibration problem, Int. J. Non-Linear Mechanics 17 (1982) 231-236.

[10] N.D. Anh, Two methods of integration of the Kolmogorov-Fokker-Planck equations (English), Ukr. Math. J. 38 (1986) 331-334; translation from Ukr. Mat. Zh. 38(3) (1986) 381-385.

[11] Z.L. Huang, W.Q. Zhu, Y. Suzuki, Stochastic averaging of strongly non-linear oscillators under combined harmonic and white noise excitations, Sound and Vibration 238 (2000) 233-256.

[12] M. Di Paola, A. Sofi, Approximate solution of the FokkerPlanckKolmogorov equation, Probabilistic Engineering Mechanics 17 (2002) 369384.

[13] Wagner UV, Wedig WV. On the calculation of stationary solution of multi-dimensional Fokker-Planck equations by orthogonal functions. Nonlinear Dynamics 21 (2000) 289-306.

[14] J.S. Yu, Y.K. Lin, Numerical path integration of a nonlinear oscillator subject to both sinusoidal and white noise excitations, Int. J. Non-Linear Mechanics 37 (2004) 1493-1500.

[15] F. Schmidt, C.H. Lamarque, Computation of the solutions of the FokkerPlanck equation for one and two DOF systems, Communications in Nonlinear Science and Numerical Simulation 14 (2009) 529-542.

[16] J. Biazar, P Gholamin, K. Hosseini, Variational iteration method for solving Fokker-Planck equation, J. of the Franklin Institute 347 (2010) 11371147.

[17] S. Narayanan, P. Kumar, Numerical solutions of Fokker-Planck equation of nonlinear systems subjected to random and harmonic excitations, Probabilistic Engineering Mechanics 27 (2012) 35-46.

[18] T.K. Caughey, Response of a nonlinear string to random loading, ASME J. of Applied Mechanics 26 (1959) 341-344.

[19] T.K. Caughey, Equivalent Linearization techniques, J. of the Acoustical Society of America 35(11) (1963) 1706-1711.

[20] T.S. Atalik, S. Utku, Stochastic of linearization of multi-degree of freedom nonlinear, J. Earth Eng Struct Dynamic 4 (1976) 441-420.

[21] L. Socha, Probability density equivalent linearization technique for nonlinear oscillator with stochastic excitations, Z. Angew. Math. Mech. 78 (1998) 1087-1088.

[22] N.D. Anh, W. Shiehlen, A technique for obtaining approximate solution in Gaussian equivalent linearization, Comput. Methods Appl. Mech. Engrg. 168 (1999) 113-119.

[23] J.B. Roberts, P.D. Spanos, Random Vibration and Statistical Linearization, Dover Publications, Inc., Mineola, New York, 1999.

[24] I. Elishakoff, L. Andrimasy, M. Dolley, Application and extension of the stochastic linearization by Anh and Di Paola, Acta Mech. 204 (2009) 89-98.

[25] N.D. Anh, N.N. Hieu, N.N. Linh, A dual criterion of equivalent linearization method for nonlinear systems subjected to random excitation, Acta Mech. 223(3) (2012) 645-654.

[26] N.D. Anh, Random oscillations in non-autonomous mechanical systems with random parametric excitation, Ukr Math J. 37 (1985) 412-416.

[27] N.D. Anh, On the study of random oscillations in non-autonomous mechanical systems using the FPK equation, Prikl. Mat. Mekh (translated from the Russian by Pergamon Journals Ltd) 49 (1985) 392-397.

[28] A.H. Nayfeh, S.J Serhan, Response statistics of nonlinear systems to combined deterministic and random excitations, Int. J. Non-linear Mechanics 25(5) (1990) 493-509.

[29] C.S. Manohar, R.N. Iyengar, Entrainment in Van der Pols oscillator in the presence of noise, Int. J. Non-linear Mechanics 26(5) (1991) 679-686.

[30] Z.L. Huang, W.Q. Zhu, Y. Suzuki, Stochastic averaging of strongly non-linear oscillators under combined harmonic and white noise excitations, Sound and Vibration 238 (2000) 233-256.

[31] J.S. Yu, Y.K. Lin YK, Numerical path integration of a nonlinear oscillator subject to both sinusoidal and white noise excitations, Int. J. Non-Linear Mechanics 37 (2004) 1493-1500.

[32] W.X. Xie, W. Xu, L. Cai, Study of the Duffing-Rayleigh oscillator subject to harmonic and stochastic excitations by path integration, Applied Mathematics and Computation 172 (2006) 1212-1224.

[33] R. Haiwu, X. Wei, M. Guang, F. Tong, Response of a Duffing oscillator to combined determinstic harmonic and random exctaion, Sound and Vibration 242(2) (2001) 362-368.

[34] N.D Anh, N.N Hieu, The Duffing oscillator under combined periodic and random excitations, Probabilistic Engineering Mechanics 30 (2012) 27-36.

[35] N.D. Anh, V.L. Zakovorotny, D.N. Hao, Response analysis of Van der Pol oscillator subjected to harmonic and random excitations. Probab Eng Mech 2014;37:51-59.