Southeast-Asian J. of Sciences: Vol. 1, No 1(2012) pp. 126-136

THE LOCATING-CHROMATIC NUMBER FOR CORONA PRODUCT OF GRAPHS

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Abstract

The locating-chromatic number of a graph G can be defined as the cardinality of a minimum ordered partition Π of the vertex set $V(G)$ such that every vertex in G has the different coordinates with respect to Π and every two adjacent vertices in G are not included in the same partition class. In this case, the coordinate of a vertex v is defined as the distances from vertex v to the ordered partition classes in Π. In this paper, we discuss the locating-chromatic number for a corona product of two graphs.

1 Introduction

The concept of graph locating-chromatic number was introduced by Chartrand, Erwin, Henning, Slater and Zhang [3] in 2002, as a marriage between two previous big concepts in graph, namely graph coloring and graph partition dimension. Let $G = (V, E)$ be a connected graph. The *distance* $d(u, v)$ between vertices u and v in G is the length of a shortest path connecting u and v in G. For $v \in V(G)$ and $S \subset E(G)$, the *distance* $d(v, S)$ from u to S is defined as $\min\{d(v, x)|x \in S\}$. In particular, if $d(x, S) \neq d(y, S)$ then we shall say that x and y are *distinguished* by S or x and y are *distinguishable*. Let c be a *proper k-coloring* of $V(G)$ which induces an ordered partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$, where S_i is the set of all vertices colored by i in G. The *color code* $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x)|x \in S_i\}$ for $1 \leq i \leq k$. If every two vertices have different

Key words: locating-chromatic number, corona graph.
2000 AMS Mathematics Subject Classification: 05C15

color codes then c is called a *locating k-coloring* of G. The *locating-chromatic number* of graph G, denoted by $\chi_L(G)$, is the smallest integer k such that G has a locating k-coloring.

The problem of determining the locating-chromatic number of any general graph is an NP-hard problem. This means that no efficient algorithm to determine the locating-chromatic number of any given graph. Therefore, some heuristics methods have been developed to determine these numbers. Some studies have also been done by applying to certain classes of graphs, such as paths, cycles, certain trees and others. Characterization studies for all graphs having a specific locating-chromatic number have also carried out. However, the results are still very limited and not yet satisfactory. Some are presented below.

Chartrand *et al.* [3] have determined the locating-chromatic numbers of some well-known classes such as paths, cycles, complete multipartite graphs and double stars. Furthermore, in [4] they showed the existence of a tree of order $n(\geq 5)$ having the locating-chromatic number k if and only if $k \in \{3, 4, ..., n-2, n\}$.

In [2], Asmiati *et al.* have managed to determine the locating-chromatic number for a special class of tree, namely an amalgamation of n stars that are not necessarily isomorphic. Furthermore, in [1], they determined the locating-chromatic number for a firecrackers graph, i.e. a special tree constructed from n stars by connecting one leaf from each star to form a path P_n.

For any given graphs G and H, define the *corona product* $G \odot H$ between G and H as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining all the vertices of the i^{th}-copy of H with the i^{th}-vertex of G. Therefore, if $V(G) = \{x_1, x_2, \cdots, x_n\}$ and $V(H) = \{a_1, a_2, \cdots, a_m\}$ then $V(G \odot H) = V(G) \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of H, and $E(G \odot H) = E(G) \cup \{x_i a_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{a_{ij} a_{ik} | 1 \leq i \leq n, 1 \leq j < k \leq m,$ whenever $a_j a_k \in E(H)\}$. For simplicity, for each i, let $A_i = \{x_i\} \cup \{a_{ij} | 1 \leq j \leq m\}$. In this paper we determine the locating-chromatic number of $G \odot H$.

The following lemma is useful in determining the locating-chromatic number of a graph G. This lemma is a modification of a similar lemma derived by Chartrand *et al.* (2003) in [5].

Lemma 1. *Let G be a connected non trivial graph. Let c be a locating coloring for G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u and v must be in different color.*

2 Results

In this section, we will present the upper bound of the locating-chromatic number for corona product of two connected graph $G \odot H$ and the exact value of the locating-chromatic number for corona product of certain graphs.

Let G and H be connected graphs. Let H_i be the i^{th}-copy of H in $G \odot H$. Because any two vertices u and v of H_i have same distance from other vertices, then by Lemma 1, we have the following lemma.

Lemma 2. *Let G and H be connected graphs. Let H_i be the i^{th}-copy of H in $G \odot H$. Then, any two vertices u and v of H_i can only be distinguished by a set R such that $R \cap V(H_i) \neq \emptyset$.*

Firstly we give the upper bound of the locating-chromatic number for corona product of two connected graphs $G \odot H$.

Theorem 1. *Let G and H be two connected graphs with diameter of $H \leq 2$ then $\chi_L(G \odot H) \leq \chi_L(G) + \chi_L(H)$.*

Proof. Let Π_G and Π_H be minimum locating colorings of G and H, respectively. Let $|V(G)| = n$. For $i = 1, 2, \cdots, n$, color/partition all the vertices of each H_i according to Π_H, say $\{V(H_i)^1, V(H_i)^2, \cdots, V(H_i)^s\}$, where $s = pd(H)$. Now, consider the coloring/partition $\Pi = \Pi_1 \cup \Pi_2$ on $G \odot H$, where $\Pi_1 = \{\cup_{i=1}^n V(H_i)^1, \cup_{i=1}^n V(H_i)^2, \cdots, \cup_{i=1}^n V(H_i)^s\}$ and $\Pi_2 = \Pi_G$. Next, we will show that Π is a locating coloring of $G \odot H$. Note that since the diameter of H is at most 2, then the distance of any two vertices $u, v \in V(H_i)$, for any i, under the corona graph $G \odot H$ is the same as its distance under the original graph H. Therefore, if the vertices $u, v \in V(H_i)$, for any i, are distinguishable by Π_H then they are distinguishable too by Π_1. Let u and v be any two vertices of $G \odot H$. If $u, v \in V(H_i)$ then they will be clearly distinguished by $\cup_{i=1}^n V(H_i)^t$ for some t. If $u, v \in V(G)$ then they will be distinguished by some set in Π_G. Now, assume that $u \in V(H_i)$ and $v \in V(G)$. If $u \in \cup_{i=1}^n V(H_i)^t$ for some t, then the distances between u and v to $\cup_{i=1}^n V(H_i)^t$ is 0 and 1, respectively. Therefore, u and v are distinguished. Now, the only case we have not considered is $u \in V(H_i)$ and $v \in V(H_j)$, for $i \neq j$. If $u, v \in \cup_{i=1}^n V(H_i)^t$ for some t then u, v are distinguished by some set in Π_G since Π_G is a locating coloring for G. $\square$

Now, we consider the corona product $G \cong P_n \odot K_m$, where P_n represents a path of order n and K_m is the complete graph on m vertices. Let $V(G) = \{x_1, x_2, \cdots, x_n\} \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of K_m. We will show that the upper bound of Theorem 1 is satisfied by $\chi_L(P_n \odot K_m)$ provided $n \geq 2(m+2)+1$.

Theorem 2. *For $n \geq 1, m \geq 2$, the locating-chromatic number of $P_n \odot K_m$ is as follows:*

$$\chi_L(P_n \odot K_m) = \begin{cases} m+1, & \textit{if } n = 1, \\ m+2, & \textit{if } 2 \leq n \leq 2(m+2), \\ m+3, & \textit{if } n \geq 2(m+2)+1. \end{cases}$$

Proof. By Lemma 1, every two vertices in $V(H_i)$ must be in different color classes. Since x_i is adjacent to all vertices of H_i, then x_i must be in a different color class other than all the color classes in $V(H_i)$. Therefore, $\chi_L(P_n \odot K_m) \geq m+1$. If $n = 1$ then it is clear that $\chi_L(P_n \odot K_m) = m+1$. If $n \geq 2$ then in order to have different color codes for each x_i we must have $\chi_L(P_n \odot K_m) \geq m+2$. Now, consider a locating $(m+2)$-coloring c on $G \cong P_n \odot K_m$, for $n, m \geq 2$. For each i, let $A_i = \{x_i\} \cup V(H_i)$. Then, we have the following facts.

Fact 1. No two integers i and j such that $c(A_i) = c(A_j)$ but $c(x_i) \neq c(x_j)$. This is true since if $j = i+1$ and $i = 1$ then x_i and a_{jk}, for some k, will have the same color codes. If $j = i+1$ and $i \neq 1$ then either the vertices (x_j and a_{ik}) or (a_{ik} and a_{jl}), will have the same color codes, for some k, l. Now, let $j \geq i+2$ and w.l.o.g. let $c(A_i) = [1, m+1]$. Then, to distinguish vertex x_i with the vertices of H_j we have that $x_{i-1} \in S_m$ or $x_{i+1} \in S_m$. Similarly, we have that $x_{j-1} \in S_m$ or $x_{j+1} \in S_m$. Therefore, there exists two vertices a_{ik} and a_{jl} with the same color codes, for some k, l.

Fact 2. No three integers i, j and k such that $c(A_i) = c(A_j) = c(A_k)$ and $c(x_i) = c(x_j) = c(x_k)$. Without loss of generality, let $c(A_i) = [1, m+1]$ and $c(x_i) = m$. Then, if $i < j < k$ then $j \geq i+2$ and $k \geq j+2$. Therefore, $d(x_i, S_m) = 1$ or 2. Similarly for x_j and x_k. This implies that the color codes of two vertices of $\{x_i, x_j, x_k\}$ will be the same, a contradiction. Therefore, this fact holds.

From these two facts, we conclude that in any locating $(m+2)$-coloring c on G, if $c(A_i) = c(A_j)$, for $i < j$, then $c(x_i) = c(x_j)$; and we cannot have three A_is with the same $c(A_i)$. Therefore, a locating $(m+2)$-coloring c can only exists on G if $n \leq 2(m+2)$. To show the coloring, let us define the mapping $c : V(P_n \odot K_m) \to [1, m+2]$ such that:

$$c(x_i) = \begin{cases} i, & \text{if } 1 \leq i \leq m+1; \\ 1, & \text{if } i = m+2; \\ m+2, & \text{if } i = m+3 \text{ or } i = m + \lceil m/2 \rceil + 4; \\ 3 + 2(i-m-4), & \text{if } m \text{ is even, } m+4 \leq i \leq m + \lceil m/2 \rceil + 3; \\ 2 + 2(i - m - \lceil m/2 \rceil - 5), & \text{if } m \text{ is even, } m + \lceil m/2 \rceil + 5 \leq i \leq n; \\ 2 + 2(i-m-4), & \text{if } m \text{ is odd, } m+4 \leq i \leq m + \lceil m/2 \rceil + 3; \\ 3 + 2(i - m - \lceil m/2 \rceil - 5), & \text{if } m \text{ is odd, } m + \lceil m/2 \rceil + 5 \leq i \leq n. \end{cases}$$

$$c(V(H_i)) = \begin{cases} [1, m+2] - \{1, m+2\}, & \text{if } i = m+3 \text{ or } i = m + \lceil m/2 \rceil + 4; \\ [1, m+2] - \{c(x_i), c(x_i)+1\}, & \text{otherwise.} \end{cases}$$

We will show that c is a locating coloring on $G \cong P_n \odot K_m$, if $2 \leq n \leq 2(m+2)$. Let u, v be two vertices in the same color class. If $u = x_i$ and $v = x_j$ for some i, j then they will have different color codes. If $u = x_i$ and $v = a_{jk}$ for some i, j, k then $1 = d(u, S_t) < d(v, S_t)$ where $t = c(j) + 1 \pmod{m+2}$. If $u = a_{ik}$ and $v = a_{jl}$ for some i, j, k, l then $d(u, S_t) \neq d(v, S_t)$ where $t = c(j) + 1 \pmod{m+2}$. Therefore c is a locating coloring on G.

Now, consider the case of $n \geq 2(m+2)+1$. To show the upper bound for this case, define the mapping $c : V(P_n \odot K_m) \rightarrow [1, m+3]$ such that:

$$c(x_i) = \begin{cases} m+3 & \text{if } i = 1, \\ 2 & \text{if } i \text{ is even}, \\ 3 & \text{if } i \text{ is odd and } i \neq 1, \end{cases}$$

$$c(V(H_i)) = \begin{cases} [1, m] & \text{if } i = 1, \\ [1, m+2] - \{2, 3\} & \text{otherwise.} \end{cases}$$

In order to show c is a locating coloring on G, we need only to consider the case of vertices u, v such that $d(u, x_1) = d(v, x_1)$. This implies that $u = a_{ik}$ and $v = x_{i+1}$ for some i, k. If $i > 1$ then u and v must be in different color classes under c. If $i = 1$ then $d(u, S_{m+1}) \neq d(v, S_{m+1})$. Therefore c is a locating coloring on G. □

Now, we consider the corona product $G \cong P_n \odot \overline{K}_m$, where P_n represents a path of order n and $\overline{K}_m$ is the complement of the complete graph on m vertices. Let the vertex-set $V(G) = \{x_i | 1 \leq i \leq n\} \cup \{a_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge-set $E(G) = \{x_{i-1}x_i | 2 \leq i \leq n\} \cup \{x_i a_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$.

Theorem 3. *For $n, m \geq 1$, the locating-chromatic number of $P_n \odot \overline{K}_m$ is as follows:*

$$\chi_L(P_n \odot \overline{K}_1) = \begin{cases} 2, & \text{if } n = 1 \\ 3, & \text{if } 2 \leq n \leq 6 \\ 4, & \text{if } n \geq 7, \end{cases}$$

$$\chi_L(P_n \odot \overline{K}_m) = \begin{cases} m+1, & \text{if } m \geq 2 \text{ and } 1 \leq n \leq m+1, \\ m+2, & \text{if } m \geq 2 \text{ and } n \geq m+2. \end{cases}$$

Proof. Let $V(P_n \odot \overline{K}_m) = \{x_1, x_2, \cdots, x_n\} \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of $\overline{K}_m$. Now, consider the following two cases.

Case 1. $m = 1$.
For $n = 1$, we have $P_1 \odot \overline{K}_1 \cong P_2$, then it is clear that $\chi_L(P_1 \odot \overline{K}_1) = 2$. Now, let $m = 1$ and $2 \leq n \leq 6$, then clearly $\chi_L(P_n \odot \overline{K}_1) \geq 3$. Figure 1 shows the locating 3-coloring for this case, therefore $\chi_L(P_n \odot \overline{K}_1) = 3$, for $2 \leq n \leq 6$.

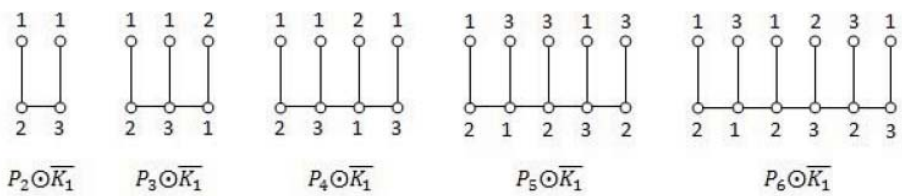

Figure 1: The locating 3-coloring of $P_n \odot \overline{K}_1$ for $2 \leq n \leq 6$.

If $n \geq 7$ then define $c : V(P_n \odot \overline{K}_1) \to [1, 4]$ such that:

$$c(x_i) = \begin{cases} 4 & \text{if } i = 1, \\ 3 & \text{if } i \text{ is odd and } i \neq 1, \\ 2 & \text{if } i \text{ is even,} \end{cases}$$

$$c(a_{i1}) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

The mapping c is a locating coloring on $G \cong P_n \odot \overline{K}_1$, since if $d(u, x_1) = d(v, x_1)$ then $u = a_{i1}$ and $v = x_{i+1}$ for some i. This implies that $u, v \in S_2$ and $1 = d(v, S_1) < d(u, S_1) = 2$. Therefore the color codes of u and v are different. To show that $\chi_L(G) \geq 4$ for $n \geq 7$, assume for a contradiction $\chi_L(G) = 3$, for $n \geq 7$. Let us call a vertex x with distance 1 to other two color classes by a *dominant vertex* in G. Then, there must be three dominant vertices in G, otherwise there will be two vertices with the same color code. These three dominant vertices must be in different color classes and they must be the vertices x_i, x_j, x_k, for some i, j, k and $1 \leq i < j < k \leq n$. Furthermore, $d(x_i, x_j)$ and $d(x_j, x_k)$ must be odd; and both are at most 3, since w.l.o.g. if $d(x_i, x_j) \geq 5$ then the color codes of x_{i+2} and $a_{(i+1)1}$ are the same, a contradiction. Next, one of these distances must be 1. Since otherwise (w.l.o.g. if $d(x_i, x_j) = d(x_j, x_k) = 3$), then either the color codes of x_{j+1} and a_{j1} are the same or the color codes of x_{j-1} and a_{j1} are the same, a contradiction. Therefore, $n < 7$, a contradiction. This concludes the proof of this case. Note that this case has been also proved in [1].

Case 2. $m \geq 2$.
Let $1 \leq n \leq m+1$. Since every two vertices a_{ij} and a_{ik}, for $1 \leq i \leq n$ and $1 \leq j < k \leq m$, we have $d(a_{ij}, x) = d(a_{ik}, x)$ for every $x \in P_n \odot \overline{K}_m - \{a_{ij}, a_{ik}\}$, then by Lemma 1, $\chi_L(P_n \odot \overline{K}_m) \geq m+1$. For the upper bound, we construct a mapping

$$c : V(P_n \odot \overline{K}_m) \to [1, m+1] \text{ such that}$$

$$c(x_i) = i, c(V(H_i)) = [1, m+1] - \{i\}, \text{for each } i.$$

Then, it is easy to see that c is a locating-coloring of $P_n \odot \overline{K}_m$, for $m \geq 2$ and $1 \leq n \leq m+1$.

Now, let $n \geq m+2$. Again, by Lemma 1, $\chi_L(P_n \odot \overline{K}_m) \geq m+1$. For a contradiction, suppose $\chi_L(P_n \odot \overline{K}_m) = m+1$. Since $n \geq m+2$, then there are two vertices $x_i, x_j \in P_n \odot \overline{K}_m$ have the same color. Since each of these two vertices adjacent to m different color vertices, then the color codes of these two vertices are the same, a contradiction. Hence, $\chi_L(P_n \odot \overline{K}_m) \geq m+2$. For the upper bound, we construct the following mapping $c : V(P_n \odot \overline{K}_m) \to [1, m+2]$:

$$c(x_i) = \begin{cases} m+2 & \text{if } i = 1, \\ m+1 & \text{if } i \text{ is odd and } i \neq 1, \\ m & \text{if } i \text{ is even,} \end{cases}$$

$$c(a_{ij}) = \begin{cases} j+1 & \text{if } i \text{ is even and } j = m, \\ j & \text{otherwise.} \end{cases}$$

To show that c is a locating coloring of $G \cong P_n \odot \overline{K}_m$, we only need to consider any two distinct vertices u and v in G satisfying $d(u, x_1) = d(v, x_1)$. Then, it implies that $u = a_{ij}$ and $v = x_{i+1}$ for some i and j. In this case, we have $1 = d(v, S_{m-1}) < d(u, S_{m-1}) = 2$. Therefore, the color codes of u and v are different. Therefore c is a locating coloring on G. □

Now, we consider the corona product $C_n \odot \overline{K}_m$, where C_n represents a cycle of order n and $\overline{K}_m$ is the complement of the complete graph on m vertices.

Theorem 4. *For $n, m \geq 1$, the locating-chromatic number of $C_n \odot \overline{K}_m$ is as follows:*

$$\chi_L(C_n \odot \overline{K}_1) = \begin{cases} 3, & \textit{if } 3 \leq n \leq 5 \\ 4, & \textit{if } n \geq 6, \end{cases}$$

$$\chi_L(C_n \odot \overline{K}_m) = \begin{cases} m+1, & \textit{if } m \geq 2 \textit{ and } 3 \leq n \leq m+1, \\ m+2, & \textit{if } m \geq 2 \textit{ and } n \geq m+2. \end{cases}$$

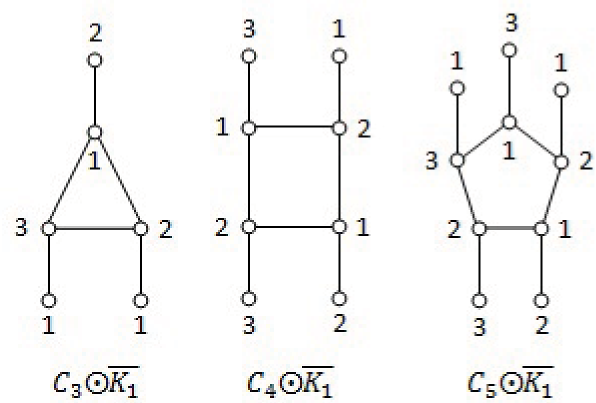

Figure 2: The locating 3-coloring of $C_n \odot \overline{K}_1$ for $3 \leq n \leq 5$.

Proof. Let $V(C_n \odot \overline{K}_m) = \{x_1, x_2, \cdots, x_n\} \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of $\overline{K}_m$. By Lemma 1, every two vertices in $V(H_i)$ must be in different color classes. Since x_i is adjacent to all vertices of H_i, then x_i must be in a different color class other than all the color classes in $V(H_i)$. Therefore, $\chi_L(C_n \odot \overline{K}_m) \geq m + 1$. Now, consider the following two cases.

Case 1. $m = 1$.
For $3 \leq n \leq 5$, then it is clear that $\chi_L(C_n \odot \overline{K}_1) \geq 3$. Figure 2 shows the locating 3-coloring for this case, therefore $\chi_L(C_n \odot \overline{K}_1) = 3$, for $3 \leq n \leq 5$.
If $n \geq 6$ then define $c : V(C_n \odot \overline{K}_1) \to [1, 4]$ such that:

$$c(x_i) = \begin{cases} 4 & \text{if } i = 1, \\ 3 & \text{if } (n \text{ and } i \text{ are odd, } i \neq 1) \text{ or} \\ & (n \text{ is even, } i \text{ is odd, } 1 < i \leq 2\lfloor \frac{n}{4} \rfloor + 1), \\ 2 & \text{if } (n \text{ is odd and } i \text{ is even}) \text{ or} \\ & (n \text{ is even, } i \text{ is odd and } i > 2\lfloor \frac{n}{4} \rfloor + 1) \text{ or} \\ & (n \text{ is even, } i \text{ is even and } i \leq \frac{n}{2}), \\ 1 & \text{if } n, i \text{ are even and } i \geq \frac{n}{2} + 1 \end{cases}$$

$$c(a_{i1}) = \begin{cases} 1 & \text{if } (n \text{ is odd and all } i) \text{ or} \\ & (n \text{ is even, } i \leq 2\lfloor \frac{n}{4} \rfloor + 1) \\ 3 & \text{if } n \text{ is even and } i > 2\lfloor \frac{n}{4} \rfloor + 1. \end{cases}$$

The mapping c is a locating coloring on $G \cong C_n \odot \overline{K}_1$, for $n \geq 6$ since if $d(u,x_1) = d(v,x_1)$ then (i) $u = a_{i1}$, $v = x_{i+1}$ for some i, (ii) $u = x_i$, $v = x_{n+1-i}$, (iii) $u = a_{i1}$, $v = x_{n+i-i}$, or (iv) $u = x_i$, $v = a_{(n+1-i)1}$. If either (i) or (ii) holds then the vertices u and v are in the different color classes. If (iii) holds then $1 = d(v,S_2) < d(u,S_2) = 2$ or u and v are in different color classes. If (iv) holds then $1 = d(u,S_2) < d(v,S_2) = 2$ or u and v are in different color classes. Therefore, in any case above the color codes of u and v are different. Therefore, c is a locating coloring.

To show that $\chi_L(G) \geq 4$ for $n \geq 6$, assume for a contradiction $\chi_L(G) = 3$, for $n \geq 6$. Recall, a vertex x is *dominant* if the distances of x to all other color classes are 1. Then, if n is odd then there are exactly three dominant vertices in G, but if n is even then there are exactly two dominant vertices. Otherwise, there will be two vertices in G with the same color code. These dominant vertices must be in different color classes and they must be the vertices x_i, x_j, x_k, for some i, j, k and $1 \leq i < j < k \leq n$. Let l_{ij}, l_{jk}, and l_{ki} be the length of shortest paths connecting x_i to x_j, x_j to x_k, or x_k to x_i, which do not pass x_k, x_i, or x_j, respectively. Then, l_{ij}, l_{jk} and l_{ki} must be odd; and these are at most 3, since without loss of generality if $l_{ij} \geq 5$ then the color codes of x_{i+2} and $a_{(i+1)1}$ are the same, a contradiction. If n is even then the distance between these two dominant vertices is either 1 or 3. If n is odd then one of l_{ij}, l_{jk} or l_{ki} is 1. Since otherwise (w.l.o.g. if $l_{ij} = l_{jk} = 3$), then either the color codes of x_{j+1} and a_{j1} are the same or the color codes of x_{j-1} and a_{j1} are the same, a contradiction. Therefore, $n \leq 7$. However, if $n = 6$ then two neighbors of the dominant vertex x_i will have the same color code. If $n = 7$ and $l_{ij} = l_{jk} = 1$ then the color codes of x_{i-2} and $a_{(i-1)1}$ are the same, a contradiction. If $n = 7$ and $l_{ij} = 1$ and $l_{jk} = 3$ then the color codes of two neighbors of the dominant vertex x_k are the same, a contradiction. Therefore, this concludes that $\chi_L(G) \geq 4$ for $n \geq 6$.

Case 2. $m > 1$.
For $3 \leq n \leq m+1$, define $c_1 : V(C_n \odot \overline{K}_m) \to [1, m+1]$ such that: $c_1(x_i) = i$ and $c_1(V(H_i)) = [1, m+1] - \{i\}$, for each i, where $c_1(V(H_i)) = \{c_1(a_{i1}), c_1(a_{i2}), \cdots, c_1(a_{im})\}$. It is clear that c_1 is a locating coloring of $C_n \odot \overline{K}_m$. Now, if $n \geq m+2$ then it is clear that $\chi_L(C_n \odot \overline{K}_m) \geq m+2$, since otherwise there are two distinct vertices x_i and x_j in the same partition class. This will imply that the color codes of these vertices are the same, a contradiction. To show the upper bound, define $c_2 : V(C_n \odot \overline{K}_m) \to [1, m+2]$ such that: $c_2(x_1) = m+2, c_2(H_1) = [1, m]$, and $c_2(x_i) = c(x_i)$, and $c_2(V(H_i)) = [1, m+1] - \{c_2(x_i)\}$, for all $i > 1$, where c is the locating coloring in Case 1. It can be verified that c_2 is a locating coloring of $C_n \odot \overline{K}_m$ for $n \geq m+2$ and $m \geq 2$. □

Now, we consider the corona product $K_n \odot \overline{K}_m$, where K_n represents a complete graph of order n and $\overline{K}_m$ is the complement of the complete graph on m vertices.

Theorem 5. *For $n, m \geq 1$, the locating-chromatic number of $K_n \odot \overline{K}_m$ is as follows:*

$$\chi_L(K_n \odot \overline{K}_m) = \begin{cases} m+1, & \text{if } n \leq m+1 \\ n, & \text{if } n > m+1 \end{cases}$$

Proof. Let $V(K_n \odot \overline{K}_m) = \{x_1, x_2, \cdots, x_n\} \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of $\overline{K}_m$. By Lemma 1, every two vertices in $V(H_i)$ must be in different color classes. Since x_i is adjacent to all vertices of H_i, then x_i must be in a different color class other than all the color classes in $V(H_i)$. Therefore, $\chi_L(K_n \odot \overline{K}_m) \geq m+1$. If $n \leq m+1$ then define a mapping c on $V(K_n \odot \overline{K}_m)$ such that $c(x_i) = i$ and $c(V(H_i)) = [1, m+1] - \{i\}$ for each i. This mapping c is a locating coloring. Therefore, $\chi_L(K_n \odot \overline{K}_m) = m+1$ for $n \leq m+1$.

Now, consider $n > m+1$. Since every two x_is must be in different classes then $\chi_L(K_n \odot \overline{K}_m) \geq n$. By constructing a mapping c such that

$$c(x_i) = i,$$

$$c(V(H_i)) = \begin{cases} [1, m+1] - \{i\} & \text{if } i \leq m+1, \\ [1, m] & \text{otherwise,} \end{cases}$$

we will show that c is a locating coloring. If $d(u, x_1) = d(v, x_1)$ then we have: (i) $u = x_k$ and $v = x_l$ for some k, l; (ii) $u = x_i$ and $v = a_{1k}$ for some $i \neq 1$ and k, or (iii) $u = a_{it}$ and $v = a_{js}$ for some i, j, t, s. If (i) holds then u and v must be in different classes. If (ii) holds then $d(u, S_{i-1}) \neq d(v, S_{i-1})$. If (iii) holds then $d(u, S_i) \neq d(v, S_i)$. Therefore, c is a locating coloring and so $\chi_L(K_n \odot K_m) = n$. □

Now, we consider the corona product $K_n \odot K_m$, where K_n represents a complete graph of order n and K_m is the complete graph on n vertices.

Theorem 6. *For $n, m \geq 1$, the locating-chromatic number of $K_n \odot K_m$ is as follows:*

$$\chi_L(K_n \odot K_m) = \begin{cases} m+1, & \text{if } n = 1 \\ m+2, & \text{if } 2 \leq n \leq m+2 \\ n, & \text{if } n > m+2. \end{cases}$$

Proof. Let $V(K_n \odot K_m) = \{x_1, x_2, \cdots, x_n\} \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_n)$, where $V(H_i) = \{a_{ij} | 1 \leq j \leq m\}$ is the vertex-set of the i^{th}-copy of K_m. By Lemma 1, every two vertices in $V(H_i)$ must be in different color classes. Furthermore, x_i is adjacent to all vertices of H_i. Then, x_i is in a different color class other than all color classes in $V(H_i)$. Therefore, $\chi_L(K_n \odot K_m) \geq m+1$. If $n = 1$ then it is clear that $\chi_L(K_n \odot K_m) = m+1$. If $2 \leq n \leq m+2$ then $\chi_L(K_n \odot K_m) \geq m+2$. Let us define a mapping c satisfying:

$$c(x_i) = i,$$

$$c(V(H_i)) = \begin{cases} [1, m+2] - \{i, i+1\} & \text{if } i \neq m+2, \\ [1, m+2] - \{1, i\} & \text{otherwise.} \end{cases}$$

Now, we will show that c is a locating coloring on $K_n \odot K_m$. If $d(u, x_1) = d(v, x_1)$ then we have: (i) $u = x_k$ and $v = x_l$ for some k, l; (ii) $u = x_i$ and $v = a_{1k}$ for some $i \neq 1$ and k, or (iii) $u = a_{it}$ and $v = a_{js}$ for some i, j, t, s. If (i) holds then u and v must be in different classes. If (ii) holds then $d(u, S_2) \neq d(v, S_2)$. If (iii) holds then $d(u, S_{i+1}) \neq d(v, S_{i+1})$ or $d(u, S_1) \neq d(v, S_1)$. Therefore, c is a locating coloring and so $\chi_L(K_n \odot K_m) = m+2$ for $2 \leq n \leq m+2$.

Next, if $n > m+2$ then it is clear that $\chi_L(K_n \odot K_m) \geq n$. Now, let us define a mapping c such that $c(x_i) = i$, and $c(V(H_i)) = X_i$, where X_i is any m-subset of $\{1, 2, \cdots, n\}$ not containing i and $i+1 \mod n$, for each i. It can be verified that c is a locating coloring. Therefore $\chi_L(K_n \odot K_m) = n$. □

Acknowledgement This research was supported by *Program Riset dan Inovasi KK - ITB* and *Riset Program Doktor Unggulan I-MHERE, FMIPA - ITB Indonesia, 2011.*

References

[1] Asmiati, E.T. Baskoro, H. Assiyatun, D. Suprijanto, R. Simanjuntak, S. Uttunggadewa, *The Locating-chromatic number of firecracker graphs,* to appear in *Far East J. Math. Sci.*

[2] Asmiati, H. Assiyatun, E.T. Baskoro, Locating-chromatic number of Amalgamation of stars, *ITB Journal of Science*, 43A:1 (2011), 1-8.

[3] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, *The locating-chromatics number of a graph,*Bull. Inst. Combin. Appl., **36** (2002), 89-101.

[4] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, *Graph of order n with locating-chromatic number n-1,* Discrete Math., **269** (2003), No.1-3, 65-79.

[5] G. Chartrand and P. Zhang, *The theory and application of resolvability in graphs,* Congressus Numerantium 160 (2003), 47-68.