***Southeast-Asian J. of Sciences: Vol. 5, No. 2 (2017)*** pp. 174-183

# PATTERN OF $(x, y, z)$ ON THE SPHERE SURFACE WITH RADIUS IS A POSITIVE INTEGER

**Boonrat Kasempitakpong** and **Suparat Leeratanawalee***

*Dept. of Mathematics, Faculty of Science and Technology,*
*Chiang Mai Rajabhat University, Chiang Mai 50300, Thailand*
*e-mail: brk34rt@hotmail.com; s-suparat@hotmail.com*

**Abstract**

For any $(x, y, z)$ in three dimensional space where $x, y, z$ are integers, there is a sphere which has a center at the origin and $(x, y, z)$ is on it's surface. It turns out that the sphere's radius need not to be an integer. The purpose of this article is to determine the pattern of such coordinate $(x, y, z)$ that resulted the sphere's radius is a positive integer. Moreover, we found some interconnection between such $(x, y, z)$ and the sphere's radius.

## 1 Introduction

Let $(x, y, z)$ be any point on the sphere's surface which has its center at the origin. In case of $x, y, z$ are integers, when we calculate for the sphere's radius $r$, it turns out that the sphere's radius need not to be an integer. Our main objective is to determine the pattern of integers $x, y, z$ where $(x, y, z)$ is a coordinate on the sphere's surface which has a center at the origin and the radius $r$ is a positive integer.

## 2 Preliminaries

We recall first some formulas which will be used throughout of this paper:

* Corresponding author.
**Key words:** sphere; origin; integer; coordinate.
2010 AMS Mathematics classification: 11D45, 14G05

(A1) $1+2+3+\ldots+n=\frac{n(n+1)}{2}; n \in \mathbb{N}$.

(A2) $1+3+5+\ldots+(2n-1)=n^2; n \in \mathbb{N}$.

(A3) For any coordinate $(a,b,c)$ on the sphere's surface which has a center at the orgin and $r$ is its radius, $a^2+b^2+c^2=r^2$.

The next theorem is the consequence of (A3).

**Theorem 2.1.** For any integers $x, y, z$, if $(x, y, z)$ is a coordinate on the sphere's surface which has a center at the origin and its radius $r$ is a positive integer, then $x \neq y \neq z$.

*Proof.* Suppose that $x=y=z$. Let $x=n$. Then $x=y=z=n$ where $n \in \mathbb{Z}$. Since $x^2+y^2+z^2=r^2, 3n^2=r^2$ and implies that $r=\sqrt{3n^2}=\sqrt{3}\,|n|$. Since $n \in \mathbb{Z}$ and $\sqrt{3} \in \mathbb{Q}', r \in \mathbb{Q}'$ which contradicts to $r \in \mathbb{Z}^+$. Thus $x \neq y \neq z$. $\square$

We set the steps to carry out the results of this article as the following:

(i) Study the basic knowledge and theorems which related to the coordinate of points on the sphere's surface and the sphere's radius.

(ii) Study the pattern of series of natural numbers which is used as a guideline to extend the concept into series of integers.

(iii) Collects the data of coordinates which was calculated the difference between the values of $x$ and $y$ into the classification table.

(iv) Analyzes the patterns which were collected and then states the hypothesis.

(v) Proves the hypothesis.

To construct the table for integers $x, y, z$ where $(x, y, z)$ is a coordinate on the sphere's surface which has a center at the origin, by Theorem 1, there has no the case that $x=y=z$. That means when we determine the $x$'s value, there exist $y, z \in \mathbb{Z}$ such that $y \neq x$ or $z \neq x$.

In this article, we let the $y$'s value is different from the $x$'s value. The $r$'s value is calculated by the formula $r=\sqrt{x^2+y^2+z^2}$ when $x, y, z$ are varied. Then we collect coordinates $(x, y, z)$ which $r$ is an integer into the classification table based on the difference between the values of $x$ and $y$. We construct a table to record values of $x, y, z$ and then analyze for the pattern which is the correlation between the values of $x, y, z$ in each row as in Table 1 where $y=x+5$.

We observe the values of $x, y, z$ from top-down of Table 1 and found that the values of $x, y, z$ arranged into sequences. The sequence of the $x$'s value is 1, 2, 3, 4, 5,.... The sequence of the $y$'s value is 6, 7, 8, 9, 10,.... The sequence of the $z$'s value is 18, 26, 36, 48, 62,.... The $n^{th}$ term of the sequence of $y$'s

Table 1:

| the $i^{th}$ row | $x$ | $y$ | $z$ | $r$ | the $i^{th}$ row | $x$ | $y$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | 18 | 19 | 9 | 9 | 14 | 138 | 139 |
| 2 | 2 | 7 | 26 | 27 | 10 | 10 | 15 | 162 | 163 |
| 3 | 3 | 8 | 36 | 37 | 11 | 11 | 16 | 188 | 189 |
| 4 | 4 | 9 | 48 | 49 | 12 | 12 | 17 | 216 | 217 |
| 5 | 5 | 10 | 62 | 63 | 13 | 13 | 18 | 246 | 247 |
| 6 | 6 | 11 | 78 | 79 | 14 | 14 | 19 | 278 | 279 |
| 7 | 7 | 12 | 96 | 97 | 15 | 15 | 20 | 312 | 313 |
| 8 | 8 | 13 | 116 | 117 | 16 | 16 | 21 | 348 | 349 |

Table 2:

| the $i^{th}$ row | $z$ | difference |
|---|---|---|
| 1 | 18 | |
| 2 | 26 | $8 = 2(4)$ |
| 3 | 36 | $10 = 2(5)$ |
| 4 | 48 | $12 = 2(6)$ |
| 5 | 62 | $14 = 2(7)$ |

value depends on the value of $x$ where $y = x+5$. For the $z$'s value, we conclude that it is changed in each row. We rearrange the value of $z$ into the table for analyze the pattern of $z$'s value as in Table 2.
row 1: $z = 18$.
row 2: $z = 26 = 18 + 8 = 18 + 2(4)$.
row 3: $z = 36 = 26 + 10 = 18 + 2(4) + 2(5)$.
row 4: $z = 48 = 36 + 12 = 18 + 2(4) + 2(5) + 2(6)$.
row 5: $z = 62 = 48+14 = 18+2(4)+2(5)+2(6)+2(7) = 18+2(4+5+6+7) = 18+2(1+2+3+4+5+6+7)-2(1+2+3) = 6+2(1+2+3+4+5+6+7)$.
...
row $n$: $z = 6 + 2(1 + 2 + 3 + \ldots + (n + 2)) = 6 + \frac{2(n+2)(n+3)}{2} = n^2 + 5n + 12$.

The next table shows some coordinates $(x, y, z)$ which give $r$ is an integer where both $x$ and $y$ are even.

There is another one pattern which related to the values of $x, y, z$ as in Table 4. row 1: $z = 16$.
row 2: $z = 28 = 16 + 12 = 16 + 4(3)$.
row 3: $z = 44 = 28 + 16 = 16 + 4(3) + 4(4)$.
row 4: $z = 64 = 44 + 20 = 16 + 4(3) + 4(4) + 4(5)$.
row 5: $z = 88 = 64+24 = 16+4(3)+4(4)+4(5)+4(6) = 16+4(3+4+5+6) =$

Table 3:

| the $i^{th}$ row | $x$ | $y$ | $z$ | $r$ | the $i^{th}$ row | $x$ | $y$ | $z$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 8 | 16 | 18 | 7 | 14 | 20 | 148 | 150 |
| 2 | 4 | 10 | 28 | 30 | 8 | 16 | 22 | 184 | 186 |
| 3 | 6 | 12 | 44 | 46 | 9 | 18 | 24 | 224 | 226 |
| 4 | 8 | 14 | 64 | 66 | 10 | 20 | 26 | 268 | 270 |
| 5 | 10 | 16 | 88 | 90 | 11 | 22 | 28 | 316 | 318 |
| 6 | 12 | 18 | 116 | 118 | 12 | 24 | 30 | 348 | 370 |

Table 4:

| the $i^{th}$ row | $z$ | difference |
|---|---|---|
| 1 | 16 | |
| 2 | 28 | $12 = 4(3)$ |
| 3 | 44 | $16 = 4(4)$ |
| 4 | 64 | $20 = 4(5)$ |
| 5 | 88 | $24 = 4(6)$ |

$16 + 4(1 + 2 + 3 + 4 + 5 + 6) - 4(1 + 2) = 4 + 4(1 + 2 + 3 + 4 + 5 + 6)$.
...
row $n$: $z_n = 4 + 4(1 + 2 + 3 + \ldots + (n + 1)) = 4 + \frac{4(n+1)(n+2)}{2} = 2n^2 + 6n + 8$.

By Table 1 and Table 3, the difference between $y$ and $x$ are 5 and 6, respectively. In order to determine the other pattern where the difference between $y$ and $x$ is an integer, we construct the table as in the same manner of the previous table and then observe and analyze for such patterns. We show the result from a pattern obtained where $y-x = 1, 2, 3, \ldots, 20$ and in case of $y = x$ also as in Table 5.

From Table 5, we see that for $y - x$ which is odd, the $x$'s value is equal to $n$. For $y - x$ which is even, $x = 2n$. We construct the data's tables for $y - x$ where $y - x$ is odd and $y - x$ is even, respectively.

Consider Table 6 when $y - x$ is odd, the $z$'s value is in the form $n^2 + bn + c$ where $b = y - x$. To determine the value of $c$, we consider the $c$'s value in each row as follows.
row 1: $c = 0 = 4(0)$.
row 2: $c = 4 = 4(1)$.
row 3: $c = 12 = 4(3) = 4(1 + 2)$.
row 4: $c = 24 = 4(6) = 4(1 + 2 + 3)$.
row 5: $c = 40 = 4(10) = 4(1 + 2 + 3 + 4)$.
...

Table 5:

| $x$ | $y$ | $z$ | $r$ | $y-x$ |
|---|---|---|---|---|
| $2n$ | $2n$ | $2n^2-1$ | $2n^2+1$ | 0 |
| $n$ | $n+1$ | $n^2+n$ | $n^2+n+1$ | 1 |
| $2n$ | $2n+2$ | $2n^2+2n$ | $2n^2+2n+2$ | 2 |
| $n$ | $n+3$ | $n^2+3n+4$ | $n^2+3n+5$ | 3 |
| $2n$ | $2n+4$ | $2n^2+4n+3$ | $2n^2+4n+5$ | 4 |
| $n$ | $n+5$ | $n^2+5n+12$ | $n^2+5n+13$ | 5 |
| $2n$ | $2n+6$ | $2n^2+6n+8$ | $2n^2+6n+10$ | 6 |
| $n$ | $n+7$ | $n^2+7n+24$ | $n^2+7n+25$ | 7 |
| $2n$ | $2n+8$ | $2n^2+8n+15$ | $2n^2+8n+17$ | 8 |
| $n$ | $n+9$ | $n^2+9n+40$ | $n^2+9n+41$ | 9 |
| $2n$ | $2n+10$ | $2n^2+10n+24$ | $2n^2+10n+26$ | 10 |
| $n$ | $n+11$ | $n^2+11n+60$ | $n^2+11n+61$ | 11 |
| $2n$ | $2n+12$ | $2n^2+12n+35$ | $2n^2+12n+37$ | 12 |
| $n$ | $n+13$ | $n^2+13n+84$ | $n^2+13n+85$ | 13 |
| $2n$ | $2n+14$ | $2n^2+14n+48$ | $2n^2+14n+50$ | 14 |
| $n$ | $n+15$ | $n^2+15n+112$ | $n^2+15n+113$ | 15 |
| $2n$ | $2n+16$ | $2n^2+16n+63$ | $2n^2+16n+65$ | 16 |
| $n$ | $n+17$ | $n^2+17n+144$ | $n^2+17n+145$ | 17 |
| $2n$ | $2n+18$ | $2n^2+18n+80$ | $2n^2+18n+82$ | 18 |
| $n$ | $n+19$ | $n^2+19n+180$ | $n^2+19n+181$ | 19 |
| $2n$ | $2n+20$ | $2n^2+20n+99$ | $2n^2+20n+101$ | 20 |

Table 6: $y-x$ is odd

| $x$ | $y$ | $z$ | $r$ | $y-x$ |
|---|---|---|---|---|
| $n$ | $n+1$ | $n^2+n$ | $n^2+n+1$ | 1 |
| $n$ | $n+3$ | $n^2+3n+4$ | $n^2+3n+5$ | 3 |
| $n$ | $n+5$ | $n^2+5n+12$ | $n^2+5n+13$ | 5 |
| $n$ | $n+7$ | $n^2+7n+24$ | $n^2+7n+25$ | 7 |
| $n$ | $n+9$ | $n^2+9n+40$ | $n^2+9n+41$ | 9 |
| $n$ | $n+11$ | $n^2+11n+60$ | $n^2+11n+61$ | 11 |
| $n$ | $n+13$ | $n^2+13n+84$ | $n^2+13n+85$ | 13 |
| $n$ | $n+15$ | $n^2+15n+112$ | $n^2+15n+113$ | 15 |
| $n$ | $n+17$ | $n^2+17n+144$ | $n^2+17n+145$ | 17 |
| $n$ | $n+19$ | $n^2+19n+180$ | $n^2+19n+181$ | 19 |

Table 7: $y-x$ is even

| $x$ | $y$ | $z$ | $r$ | $y-x$ |
|---|---|---|---|---|
| $2n$ | $2n$ | $2n^2-1$ | $2n^2+1$ | 0 |
| $2n$ | $2n+2$ | $2n^2+2n$ | $2n^2+2n+2$ | 2 |
| $2n$ | $2n+4$ | $2n^2+4n+3$ | $2n^2+4n+5$ | 4 |
| $2n$ | $2n+6$ | $2n^2+6n+8$ | $2n^2+6n+10$ | 6 |
| $2n$ | $2n+8$ | $2n^2+8n+15$ | $2n^2+8n+17$ | 8 |
| $2n$ | $2n+10$ | $2n^2+10n+24$ | $2n^2+10n+26$ | 10 |
| $2n$ | $2n+12$ | $2n^2+12n+35$ | $2n^2+12n+37$ | 12 |
| $2n$ | $2n+14$ | $2n^2+14n+48$ | $2n^2+14n+50$ | 14 |
| $2n$ | $2n+16$ | $2n^2+16n+63$ | $2n^2+16n+65$ | 16 |
| $2n$ | $2n+18$ | $2n^2+18n+80$ | $2n^2+18n+82$ | 18 |
| $2n$ | $2n+20$ | $2n^2+20n+99$ | $2n^2+20n+101$ | 20 |

row $k$: $c = 4(1+2+3+\ldots+(k-1)) = 2k(k-1)$.

Consider table 7 when $y-x$ is even, the $z$'s value is in the form $2n^2+bn+c$ where $b = y-x$. To determine the value of $c$, we consider the $c$'s value in each row as follows.
row 1: $c = -1$.
row 2: $c = 0$.
row 3: $c = 3$.
row 4: $c = 8$.
row 5: $c = 15$..
....
Notice that the $c$'s value in row 3, 4 and 5 are related to the order of rows as follows.
row 3: $c = 3 = 2^2-1 = (3-1)^2-1$.
row 4: $c = 8 = 3^2-1 = (4-1)^2-1$.
row 5: $c = 15 = 4^2-1 = (5-1)^2-1$.

By the previous patterns from our observation, we turn back to row 1 and row 2 then we get the values of $c$ as follows.
row 1: $c = -1 = 0^2-1 = (1-1)^2-1$.
row 2: $c = 0 = 1^2-1 = (2-1)^2-1$.
row 3: $c = 3 = 2^2-1 = (3-1)^2-1$.
row 4: $c = 8 = 3^2-1 = (4-1)^2-1$.
row 5: $c = 15 = 4^2-1 = (5-1)^2-1$.
And then we have
row $k$: $c = (k-1)^2-1$.

## 3 Main Results

From the research, we get the condition of $x, y, z$ where $(x, y, z)$ is on the sphere's surface as in the following theorem.

**Theorem 3.1.** For any coordinate $(x, y, z)$ on the sphere's surface which has a center at the origin, the sphere's radius is an integer if the values of $x, y, z$ satisfy with the following cases.
Case 1: $y - x$ is odd, $x = n, y = n + (2k - 1), z = n^2 + n(2k - 1) + 2k(k - 1)$.
Case 2: $y-x$ is even, $x = 2n, y = 2n+(2k-1), z = 2n^2+2n(k-1)+(k-1)^2-1$.
Where $n \in \{1, 2, 3, \ldots\}$ and $k \in \{1, 2, 3, \ldots\}$.

*Proof.* Let $P(x, y, z)$ be a point on the sphere's surface with has a center at the origin.
Case 1: $y - x$ is odd, $x = n, y = n + (2k - 1), z = n^2 + n(2k - 1) + 2k(k - 1)$.
Consider

$$\begin{aligned}
r^2 &= x^2 + y^2 + z^2 \\
&= n^2 + (n + (2k - 1))^2 + (n^2 + n(2k - 1) + 2k(k - 1))^2 \\
&= n^2 + n^2 + 2n(2k - 1) + (2k - 1)^2 + n^4 + n^2(2k - 1)^2 + 4k^2(k - 1)^2 \\
&\quad +2n^3(2k - 1) + 4n^2k(k - 1) + 4nk(2k - 1)(k - 1) \\
&= (n^4 + 2n^3(2k - 1) + n^2(2k - 1)^2) \\
&\quad +[2n^2 + 4n^2k(k - 1) + 2n(2k - 1) + 4nk(2k - 1)(k - 1)] \\
&\quad +((2k - 1)^2 + 4k^2(k - 1)^2) \\
&= (n^2 + n(2k - 1))^2 + 2[n^2 + 2n^2k(k - 1) + n(2k - 1) + 2nk(2k - 1)(k - 1)] \\
&\quad +(4k^4 - 8k^3 + 8k^2 - 4k + 1) \\
&= (n^2 + n(2k - 1))^2 + 2[n^2(1 + 2k(k - 1)) + n(2k - 1)(1 + 2k(k - 1))] \\
&\quad +(2k^2 - 2k + 1)^2 \\
&= (n^2 + n(2k - 1))^2 + 2[n^2(2k^2 - 2k + 1) + n(2k - 1)(2k^2 - 2k + 1)] \\
&\quad +(2k^2 - 2k + 1)^2 \\
&= (n^2 + n(2k - 1))^2 + 2(n^2 + n(2k - 1))(2k^2 - 2k + 1) + (2k^2 - 2k + 1)^2.
\end{aligned}$$

Thus $r^2 = ((n^2 + n(2k - 1)) + (2k^2 - 2k + 1))^2$. Then we have $r = n^2 + n(2k - 1) + (2k^2 - 2k + 1)$ is an integer because $n$ and $k$ are integers.
Case 2: $y-x$ is even, $x = 2n, y = 2n+2(k-1), z = 2n^2+2n(k-1)+(k-1)^2-1$.
By the same manner as in Case 1, we have $r^2 = (2n^2+2n(k-1)+k^2-2k+2)^2$.
Then $r = 2n^2 + 2n(k - 1) + k^2 - 2k + 2$ is an integer. □

Furthermore, we found the pattern of coordinate $(x, y, z)$ on the sphere's surface which has a center at the origin when $x, y, z$ are integers and $r$ is a positive integer as in the following theorem.

**Theorem 3.2.** For any coordinate $(x, y, z)$ on the sphere's surface which has a center at the origin and a radius $r$ where $x, y, z$ are integers, if $x$ and $y$ are odd then $r$ is not an integer.

*Proof.* Let $r$ be an integer. Since $(x, y, z)$ is on the sphere's surface which has a center at the origin and a radius $r$, so $x^2 + y^2 + z^2 = r^2$ and then $r^2 - z^2 = x^2 + y^2 > 0$. Since $x, y$ are odd, so $x = 2m+1$ and $y = 2n+1$ for some $m, n \in \mathbb{Z}$. Then $r^2 - z^2 = (2m+1)^2 + (2n+1)^2 = 2(2m^2+2m+2n^2+2n+1) = 2w$ where $w = 2m^2+2m+2n^2+2n+1 \in \mathbb{Z}$. That means $(r-z)(r+z) = 2w$. Since $r$ and $z$ are integers, we conclude that $r - z$ and $r + z$ are integers. Consider $(r - z)(r + z) = 2w$.
Case 1: $w$ is a prime number.
Case 1.1: $r - z = 1$. By $(r - z)(r + z) = 2w$, we have $r + z = 2w$. Thus $2r = 1 + 2w$ and implies that $r = \frac{1}{2} + w \notin \mathbb{Z}$ which contradicts to $r$ is an integer.
Case 1.2: $r - z = 2$. Then $r + z = w$ and $2r = 2 + w$. That means $r = 1 + \frac{w}{2}$. Since $w$ is odd, so $r = 1 + \frac{w}{2} \notin \mathbb{Z}$ which contradicts to $r$ is an integer.
Case 1.3: $r - z = w$. Then $r + z = 2$. If $r - z = 2w$, we have $r + z = 1$ and then $r$ is not an integer, which is a contradiction.
Case 2: $w$ is not a prime number. Let $w = p_1p_2p_3 \cdots p_n$ where $p_i$ is a prime number for every $i = 1, 2, 3, \ldots, n$ and $p_i, p_j$ can be the same for $i \neq j$. We set $p_1p_2p_3 \cdots p_n = ab$ where $a, b$ are prime numbers or in the form of the product of prime numbers which can be repeated or non-repeated. Then $(r - z)(r + z) = 2w = 2(ab) = (2a)b = a(2b)$. Let $r - z = 2a$. Then $r + z = b$. Thus $2r = 2a + b$ and implies that $r = a + \frac{b}{2} \notin \mathbb{Z}$. Moreover if we let $r - z = 2b$ and $r + z = a$, by the same manner as above we have $r \notin \mathbb{Z}$ which is also contradicts to $r$ is an integer. Furthermore, in case of $x$ and $z$ are odd or $y$ and $z$ are odd, we can prove in the similar way and then we have also a contradiction. □

**Theorem 3.3.** For the sphere which has a center at the origin and a radius $r$, if $r$ is even and $x, y, z$ are integers then $x, y, z$ are even.

*Proof.* Let $S$ be a sphere which has a center at the origin and a radius $r$ where $r$ is even and $x, y, z$ are integers. Since $r \in \mathbb{Z}$, by Theorem 3.2 we have that for any coordinate $(x, y, z)$ on the sphere's surface there exists not more than one of $x, y, z$ is odd. Since $r$ is even, so in case of there is only one of $x, y, z$ is odd we get $x^2 + y^2 + z^2$ is odd, which contradicts to $r$ is even. Thus $x, y, z$ are even. □

**Theorem 3.4.** For the sphere which has a center at the origin and a radius $2^n$ where $n$ is a positive integer, if $(x, y, z)$ is on the sphere's surface where $x, y, z$ are integers, then there is only one of $x, y, z$ is non-zero.

*Proof.* Let $S$ be a sphere which has a center at the origin and a radius $r = 2^n$ where $n$ a positive integer and let $(x, y, z)$ is on the surface of $S$.

Case 1: $x, y, z$ are integers which are non-zero. Consider $x^2 + y^2 + z^2 = r^2 = (2^n)^2$. Since $n \in \mathbb{Z}^+$, so $r$ is even. By Theorem 3.3, we have $x, y, z$ are even. Let $x = 2x_1, y = 2y_1$ and $z = 2z_1$ for some $x_1, y_1, z_1 \in \mathbb{Z}$. By $x^2+y^2+z^2 = r^2 = (2^n)^2$, we have $(2x_1)^2+(2y_1)^2+(2z_1)^2 = (2^n)^2$ and implies that $4x_1^2+4y_1^2+4z_1^2 = 2^{2n} = 4^n$. Then $x_1^2+y_1^2+z_1^2 = 4^{n-1} = 2^{2(n-1)} = (2^{n-1})^2$. In case of $n \geq 1$ we have $2^{n-1}$ is even and then $x_1, y_1, z_1$ are even. Let $x_1 = 2x_2, y_1 = 2y_2$ and $z_1 = 2z_2$ for some $x_2, y_2, z_2 \in \mathbb{Z}$. By $x_1^2+y_1^2+z_1^2 = 2^{2(n-1)} = (2^{n-1})^2$, we have $(2x_2)^2+(2y_2)^2+(2z_2)^2 = 2^{2(n-1)}$ and then $4x_2^2+4y_2^2+4z_2^2 = 2^{2(n-1)} = 4^{n-1}$ implies that $x_2^2 + y_2^2 + z_2^2 = 4^{n-2} = 2^{2(n-2)} = (2^{n-2})^2$. Let $x_2 = 2x_3, y_2 = 2y_3, z_2 = 2z_3$ for some $x_3, y_3, z_3 \in \mathbb{Z}$. Then $x_3^2+y_3^2+z_3^2 = 4^{n-3} = 2^{2(n-3)} = (2^{n-3})^2$. We do this in the same way until we get $x_k^2 + y_k^2 + z_k^2 = 4^{n-k} = 2^{2(n-k)}$ where $k = n-1$. Then $x_{n-1}^2 + y_{n-1}^2 + z_{n-1}^2 = 2^{2(1)} = 2^2 = 4$ and $x_{n-1}, y_{n-1}, z_{n-1}$ are even which are non-zero. The consequence is $x_{n-1}^2 \geq 4, y_{n-1}^2 \geq 4, z_{n-1}^2 \geq 4$. Then $x_{n-1}^2 + y_{n-1}^2 + z_{n-1}^2 \geq 12$ which contradicts to $x_{n-1}^2+y_{n-1}^2+z_{n-1}^2 = 4$. So there has no $(x, y, z)$ on the sphere's surface which $x, y, z$ are non-zero.
Case 2: $x$ or $y$ or $z$ only one zero. Since 0 is even, then the proof is similar to the proof of Case 1 and we conclude that there has no $(x, y, z)$ on the sphere's surface which has a center at the origin and a radius is an integer where $x$ or $y$ or $z$ only one zero.
Case 3: $x$ or $y$ or $z$ only one non-zero. Let $x \neq 0$ then $y = z = 0$. Since $x^2+y^2+z^2 = r^2 = (2^n)^2, x^2+0^2+0^2 = r^2 = (2^n)^2$ and implies that $x = \pm 2^n$. Thus $(0, y, 0)$ and $(0, 0, z)$ are on the sphere's surface where $y = \pm 2^n$ and $z = \pm 2^n$ respectively.
Case 4: $x = y = z = 0$. Since $r$ is a positive integer, it cannot be happened that $x^2 + y^2 + z^2 = r^2$ where $x = y = z = 0$. By the four cases, we have that if $(x, y, z)$ is on the sphere's surface where $x, y, z$ are integers, then there exists $x$ or $y$ or $z$ only one non-zero. □

# 4 Conclusions and discussion

For any coordinate $(x, y, z)$, when switching between the first, the second or the third coordinates we will get the different point from the original but it still be a coordinate on the sphere. The condition of coordinate $(x, y, z)$ which is on the sphere's surface where has a center at the origin and a radius is a positive integer allowing us identify the position of a point on the sphere's surface easily when $x, y$ are arbitrary integers such that $x \neq y$. We have $z \in \mathbb{Z}$ and $r \in \mathbb{Z}^+$ where the value of $z$ depends on the values of $x$ and $y$. The length of a radius $r$ will not change when switching between $x$ or $y$ or $z$.

Moreover, we found the patterns of coordinate $(x, y, z)$ on the sphere's surface which has a center at the origin where $x, y, z$ are integers and a radius $r$ is a positive integer such that we can observe by its summarized as follows.

(1) There has no point which the first, the second and the third coordinates are the same, i.e. there has no coordinate $(a, a, a)$ on the sphere's surface where $a$ is an integer.

(2) If $(x, y, z)$ is on the sphere's surface, there exists not more than one of $x, y, z$ is odd. The convert of this statement is not true as the following example. Let $(3, 6, 8)$ be a coordinate on a sphere's surface. Then the sphere's radius $r = \sqrt{109}$ which is not an integer.

(3) If the radius $r$ is even, each of the first, the second and the third coordinates are even.

(4) If the radius $r = 2^n$, there is only one of the first, the second and the third coordinates is non-zero.

Suggestion: The coordinate $(x, y, z)$ on the sphere's surface which has a center at the origin and a radius is a positive integer may be found in other patterns.

# References

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