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# ON THE EXPONENTIAL DIOPHANTINE EQUATION $2^x - 3^y = z^2$

**Sutthiwat Thongnak, Wariam Chuayjan**
and
**Theeradach Kaewong**

*Department of Mathematics and Statistics,*
*Faculty of Science, Thaksin University,*
*Phatthalung, 93210, Thailand*
*email: t.sutthiwat@gmail.com*

**Abstract**

In this paper, we prove the solutions of the exponential Diophantine equation $2^x - 3^y = z^2$ where $x, y$ and $z$ are non-negative integers. To find the solution, Catalan 's conjecture and division algorithm congruence were applied. The result indicates that the equation has three solutions $(x, y, z)$ including $(0, 0, 0), (1, 0, 1)$ and $(2, 1, 1)$.

## 1 Introduction

Over a decade, several mathematical researches have investigated the solution of the exponential Diophantine equation of the form $a^x + b^y = z^2$ with given constant $a$ and $b$ where $x, y$ and $z$ are non-negative integers. Because there was no general theory of the exponential Diophantine equation, a number of the equations were solved via a variety of methods. In 2007, Acu [1] solved the equation for $a = 2$ and $b = 5$. The non-negative integer solutions to the equation are $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. During 2008 - 2016, the researches related exponential diophantion appears in [5], [9]-[10], [11]-[22]. Recently, Jayakumar and Shankaralidoss [6] study solution of the Diophantine equation $47^x + 2^y = z^2$. They proved that there is a unique non-negative solution $(x, y, z) \in \{(0, 3, 3)\}$ to the equation. More researches on the Diophentine equation released in 2017 appeared in [2]-[4], [7]. However, there are still more exponential Diophantine equations that we need to prove their solutions.

**Key words:** exponential Diophantine equation, integer solution, congruence
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In this paper, we solve a new exponential Diophentine equation of the form $a^x - b^y = z^2$ with $a = 2$ and $b = 3$ where $x, y$ and $z$ are non-negative integers.

# 2 Preliminaries

**Proposition 2.1.** *[8] **(Catalan 's conjecture)** $(3, 2, 2, 3)$ is a unique solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a, b, x$ and $y$ are integers such that $min\{a, b, x, y\} > 1$.*

# 3 Main Result

**Theorem 3.1.** *The Diophantine equation $2^x - 3^y = z^2$ has three non-negative integer solutions $(x, y, z)$ including $(0, 0, 0), (1, 0, 1)$ and $(2, 1, 1)$.*

**Proof.** We suppose $x, y$ and $z$ to be non-negative integers such that $2^x - 3^y = z^2$. To find the solutions, we consider the variable $x$ into two cases.

**Case** $x = 0$. We have $1 - 3^y = z^2$. If $y = 0$, then we have $z^2 = 0$. Hence, a solution $(x, y, z)$ is $(0, 0, 0)$. If $y \geq 1$, then we have $z^2 \leq -2$ which is impossible.

**Case** $x \geq 1$. In this case, we divide number $y$ into two subcases.

**Subcase** $y = 0$. We have $2^x - z^2 = 1$. From Catalan 's Conjecture, the equation has no solutions when $x$ and $z > 1$. It is sufficient to consider in the cases $x = 1$ or $z \leq 1$. For $x = 1$, we have $2 - z^2 = 1$. It is simple to obtain $z = 1$. It follows that $(x, y, z)$ =$(1, 0, 1)$. For $z \leq 1$, it is obvious to consider $z = 0$ and $z = 1$. If $z = 0$, we obtain $2^x = 1$. Then $x = 0$. If $z = 1$, it is easy to obtain that $x = 1$. A solution $(x, y, z)$ is $(1, 0, 1)$.

**Subcase** $y \geq 1$. In this subcase, we consider for $x = 1$ or $x \geq 2$.

If $x = 1$, then it follows that $z^2 \leq -1$. This is impossible.

If $x \geq 2$, we divide number $x$ into even and odd.

For $x$ is even, there is a positive integer $k$ such that $x = 2k$, $\exists k \geq 1$. From $2^x - 3^y = z^2$, we have

$$2^{2k} - z^2 = 3^y. \tag{3.1}$$

It follows that $3^y = 2^{2k} - z^2 = (2^k - z)(2^k + z)$. Let $p + q = y$ where $p$ and $q$ are non-negative integers and $0 \leq p < q$. Thus, we have

$$2^k - z = 3^p, \tag{3.2}$$

$$2^k + z = 3^q. \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$2^{k+1} = 3^p(1 + 3^{q-p}). \tag{3.4}$$

This implies that $3^p | 2^{k+1}$. Thus $p = 0$. From (3.4), we get

$$2^{k+1} - 3^q = 1. \tag{3.5}$$

By proposition 2.1, it is sufficient to consider $k+1 \leq 1$ or $q \leq 1$. For $k+1 \leq 1$, we obtain that $k \leq 0$ which is contradiction because $k \geq 1$. For $q \leq 1$, this impiles that $q = 1$. From (3.5), then $2^{k+1} = 4$. We obtain $k = 1$ so $x = 2$. From (3.3), we have $z = 1$. Since $0 \leq p < q = 1$. This impiles that $p = 0$. We get $y = p + q = 0 + 1 = 1$. Thus, another solution is $(x, y, z) = (2, 1, 1)$. For $x$ is odd, we have $2^{2k+1} - z^2 = 3^y$ where $k$ is a positive integer. Then we have $2^{2k+1} - 3^y = z^2$. Since $2^{2k+1} \equiv -1 \,(mod\, 3)$ and $3^y \equiv 0 \,(mod\, 3)$, it follows that $z^2 \equiv -1 \,(mod\, 3)$. This is a contradiction because $z$ is integer. □

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