

ON THE EXPONENTIAL DIOPHANTINE EQUATION $2^x - 3^y = z^2$

Sutthiwat Thongnak, Wariam Chuayjan
and
Theeradach Kaewong

*Department of Mathematics and Statistics,
Faculty of Science, Thaksin University,
Phatthalung, 93210, Thailand
email: t.sutthiwat@gmail.com*

Abstract

In this paper, we prove the solutions of the exponential Diophantine equation $2^x - 3^y = z^2$ where x, y and z are non-negative integers. To find the solution, Catalan's conjecture and division algorithm congruence were applied. The result indicates that the equation has three solutions (x, y, z) including $(0, 0, 0)$, $(1, 0, 1)$ and $(2, 1, 1)$.

1 Introduction

Over a decade, several mathematical researches have investigated the solution of the exponential Diophantine equation of the form $a^x + b^y = z^2$ with given constant a and b where x, y and z are non-negative integers. Because there was no general theory of the exponential Diophantine equation, a number of the equations were solved via a variety of methods. In 2007, Acu [1] solved the equation for $a = 2$ and $b = 5$. The non-negative integer solutions to the equation are $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. During 2008 - 2016, the researches related exponential diophantion appears in [5], [9]-[10], [11]-[22]. Recently, Jayakumar and Shankaralidoss [6] study solution of the Diophantine equation $47^x + 2^y = z^2$. They proved that there is a unique non-negative solution $(x, y, z) \in \{(0, 3, 3)\}$ to the equation. More researches on the Diophantine equation released in 2017 appeared in [2]-[4], [7]. However, there are still more exponential Diophantine equations that we need to prove their solutions.

Key words: exponential Diophantine equation, integer solution, congruence
2010 AMS Mathematics Subject Classification: 11D61.

In this paper, we solve a new exponential Diophantine equation of the form $a^x - b^y = z^2$ with $a = 2$ and $b = 3$ where x, y and z are non-negative integers.

2 Preliminaries

Proposition 2.1. [8] (*Catalan's conjecture*) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

3 Main Result

Theorem 3.1. The Diophantine equation $2^x - 3^y = z^2$ has three non-negative integer solutions (x, y, z) including $(0, 0, 0)$, $(1, 0, 1)$ and $(2, 1, 1)$.

Proof. We suppose x, y and z to be non-negative integers such that $2^x - 3^y = z^2$. To find the solutions, we consider the variable x into two cases.

Case $x = 0$. We have $1 - 3^y = z^2$. If $y = 0$, then we have $z^2 = 0$. Hence, a solution (x, y, z) is $(0, 0, 0)$. If $y \geq 1$, then we have $z^2 \leq -2$ which is impossible.

Case $x \geq 1$. In this case, we divide number y into two subcases.

Subcase $y = 0$. We have $2^x - z^2 = 1$. From Catalan's Conjecture, the equation has no solutions when x and $z > 1$. It is sufficient to consider in the cases $x = 1$ or $z \leq 1$. For $x = 1$, we have $2 - z^2 = 1$. It is simple to obtain $z = 1$. It follows that $(x, y, z) = (1, 0, 1)$. For $z \leq 1$, it is obvious to consider $z = 0$ and $z = 1$. If $z = 0$, we obtain $2^x = 1$. Then $x = 0$. If $z = 1$, it is easy to obtain that $x = 1$. A solution (x, y, z) is $(1, 0, 1)$.

Subcase $y \geq 1$. In this subcase, we consider for $x = 1$ or $x \geq 2$.

If $x = 1$, then it follows that $z^2 \leq -1$. This is impossible.

If $x \geq 2$, we divide number x into even and odd.

For x is even, there is a positive integer k such that $x = 2k$, $\exists k \geq 1$. From $2^x - 3^y = z^2$, we have

$$2^{2k} - z^2 = 3^y. \quad (3.1)$$

It follows that $3^y = 2^{2k} - z^2 = (2^k - z)(2^k + z)$. Let $p + q = y$ where p and q are non-negative integers and $0 \leq p < q$. Thus, we have

$$2^k - z = 3^p, \quad (3.2)$$

$$2^k + z = 3^q. \quad (3.3)$$

From (3.2) and (3.3), we obtain

$$2^{k+1} = 3^p(1 + 3^{q-p}). \quad (3.4)$$

This implies that $3^p | 2^{k+1}$. Thus $p = 0$. From (3.4), we get

$$2^{k+1} - 3^q = 1. \quad (3.5)$$

By proposition 2.1, it is sufficient to consider $k+1 \leq 1$ or $q \leq 1$. For $k+1 \leq 1$, we obtain that $k \leq 0$ which is contradiction because $k \geq 1$. For $q \leq 1$, this implies that $q = 1$. From (3.5), then $2^{k+1} = 4$. We obtain $k = 1$ so $x = 2$. From (3.3), we have $z = 1$. Since $0 \leq p < q = 1$. This implies that $p = 0$. We get $y = p + q = 0 + 1 = 1$. Thus, another solution is $(x, y, z) = (2, 1, 1)$. For x is odd, we have $2^{2k+1} - z^2 = 3^y$ where k is a positive integer. Then we have $2^{2k+1} - 3^y = z^2$. Since $2^{2k+1} \equiv -1 \pmod{3}$ and $3^y \equiv 0 \pmod{3}$, it follows that $z^2 \equiv -1 \pmod{3}$. This is a contradiction because z is integer. $\square$

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