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THE CONDITIONS OF SOME CAYLEY DIGRAPHS CONTAINING HAMILTONIAN PATH AND HAMILTONIAN CIRCUIT

Ng. Danpattanamongkon, S. Meechana
and
D. Samana,

Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang
Ladkrabang, Bangkok, Thailand.
email: dechasamana@hotmail.com

Abstract

For a finite semigroup S and a nonempty subset A of S the *Cayley Digraph* of S with respect to A, denoted by $Cay(S, A)$ is the directed graph with vertex set S and arc set $\{(s, sa) \mid s \in S \text{ and } a \in A\}$. For digraph D, a directed path and a directed circuit which contain every vertex of D is called a *Hamiltonian path* and *Hamiltonian circuit*, respectively. In this paper, we obtain some necessary and sufficient conditions of S and A such that $|A| \leq 2$ that $Cay(S, A)$ contain a Hamiltonian circuit and a Hamiltonian path.

1 Introduction

A *digraph* $D = (V, E)$ is defined by a set V of vertices and a set E of arcs. For vertices u_1 and u_k in digraph D, a $u_1 - u_k$ *directed walk* in D is an alternating sequence $u_1, e_1, u_2, ..., e_{k-1}, u_k$ of vertices and arcs, beginning with u_1 and ending with u_k, such that $e_i = (u_i, u_{i+1})$ for $i = 1, 2, ..., k-1$. If the vertices $u_1, u_2, ..., u_k$ are distinct, then the $u_1 - u_k$ directed walk is $u_1 - u_k$ *directed path.* If $u_1 = u_k$, where $k \geq 3$, and the vertices $u_1, u_2, ..., u_k$ are distinct, then the directed walk is called a *directed circuit.* A digraph D is *strongly connected* if for every pair u, v of vertices, D contains both a $u - v$

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directed path and a $v-u$ directed path. A directed path and a directed circuit in D are called a *Hamiltonian path* and *Hamiltonian circuit*, respectively if contain every vertex of D.

Hamiltonian paths and Hamiltonian circuits are the mathematical field of graph theory which are interested. The problem of finding the Hamiltonian path and the Hamiltonian circuit are NP-complete. They can apply to many problems, such as, pizza delivery, mail delivery, traveling sales man, garbage pick up, bus service/limosine service and reading gas meters. Furthermore, they have been applied to biology[1].

In 1878, Arthur Cayley introduced definition of the Cayley digraph of group. For G is a finite group and A is a set of generators of G. The Cayley digraph of G with respect to A, denoted by $Cay(G,A)$ to be the directed graph with vertex set G and arc set $\{(g,ga) : g \in G \text{ and } a \in A\}$. Moreover, Cayley studied properties of a group such as commutativity and the multiplication table can be recovered from the Cayley digraph. A well known conjecture of Lovasz states[14] that every Cayley digraph is a Hamiltonian path. There are many researchers that studied about this conjecture. In 1976, Nathanson[15] said that the finite group G is generated by two element a and b, such that $a^2 = b^3 = e$. If $|G| \geqslant 9|ab^2|$, then the Cayley digraph $Cay(G,\{a,b\})$ does not have a Hamiltonian path. In 1978, Holszyński and Strube[20] showed that every connected Cayley digraph on any abelian group has a Hamiltonian path. In 1982, Dragon[8] proved that if either G is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian of odd order, then every connected Cayley digraph of G is hamiltonian. In 1983, Marušić[5] showed that if either G is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian group of odd order, then every connected Cayley digraph of G is Hamiltonian circuit. In 2009, Pak and Radoičić[11] said that every finite group G of size $|G| \geq 3$ has a generating set A of size $|A| \leq log_2|G|$, such that the corresponding Cayley digraph (G,A) contains a Hamiltonian circuit. In 2015, Suksumran and Panma proved Theorem 1.1, the digraph is strongly connected, this means the digraph may have a Hamiltonian path.

Theorem 1.1. *[19] S is a right simple semigroup and $A \subseteq S$. A Cayley digraph $Cay(S,A)$ is strongly connected if and only if $S = \langle A\rangle$.*

In 2016, Khosravi [2] said that if $Cay(S,A)$ is strongly connected, then $S = \langle A\rangle$. Hence, we will consider some necessary and sufficient conditions of S and A that $Cay(S,A)$ has Hamiltonian paths or Hamiltonian circuits.

2 Main Results

First, we will show the conditions of S and A that $Cay(S,A)$ have a Hamiltonian paths or Hamiltonian circuits when $|A| = 1$.

Theorem 2.1. *S is a cyclic semigroup generated by a if and only if $Cay(S,\{a\})$ contain a Hamiltonian path.*

Proof. Let S be a cyclic semigroup generated by a with order k. Then $S = \{a, a^2, a^3, \ldots, a^k\}$ and $a^i \neq a^j$ for all $i, j \in \{1, 2, 3, \ldots, k\}$ which $i \neq j$. Since $a \cdot a = a^2$, vertex a and vertex a^2 are adjacent in $Cay(S,\{a\})$. Since $a^2 \cdot a = a^3$, vertex a^2 and vertex a^3 are adjacent in $Cay(S,\{a\})$. Continue this process, we have a directed path from a to a^k. That means $Cay(S,\{a\})$ contain a Hamiltonian path. Conversely, assume that there is $a \in S$ such that $Cay(S,\{a\})$ containing a Hamiltonian path say H. Suppose that $S \neq \langle a \rangle$. Hence there exits $a_0 \in \langle a \rangle$ and $b \in S \setminus \langle a \rangle$ such that vertex a_0 adjacent to vertex b in directed path H. This implies $b = a_0 \cdot a \in \langle a \rangle$, contradiction. Therefore S is a cyclic semigroup generated by a. $\square$

Theorem 2.2. *S is a right simple cyclic semigroup generated by a if and only if $Cay(S,\{a\})$ contain a Hamiltonian circuit.*

Proof. Let S be a right simple cyclic semigroup generated by a with order k. By the proof of Theorem 2.1, we have a Hamiltonian path $a, a^2, a^3, \ldots, a^k$ in $Cay(S,\{a\})$. By Theorem 1.1, we have that $Cay(S,\{a\})$ is a strongly connected digraph. Hence the vertex a^k must have a directed path to vertex a. This shows, $Cay(S,\{a\})$ contain a Hamiltonian circuit. Conversely, assume that there is $a \in S$ such that $Cay(S,\{a\})$ contain a Hamiltonian circuit length k. Then $Cay(S,\{a\})$ contain a Hamiltonian path length k. By Theorem 2.1, $S = \langle a \rangle = \{a, a^2, a^3, \ldots, a^k\}$ and $a^{k+1} = a$. Let $b \in S$. Hence $b = a^m$ for some $m \in \{1, 2, 3, \ldots, k\}$ so $b \cdot S = a^m \cdot \{a, a^2, a^3, \ldots, a^k\} = \{a^{m+1}, a^{m+2}, a^{m+3}, \ldots, a^{m+k}\} = S$. This shows that $b \cdot S = S$ for all $b \in S$. Therefore S is a right simple cyclic semigroup generated by a. $\square$

Notice that for any semigroup S and $a \in S$ with $a \neq b$, it is easy to see that $Cay(S,\{a\})$ is a spanning subgraph of $Cay(S,\{a,b\})$. Hence a Hamiltonian path of $Cay(S,\{a\})$ is a Hamiltonian path of $Cay(S,\{a,b\})$ and a Hamiltonian circuit of $Cay(S,\{a\})$ is a Hamiltonian circuit of $Cay(S,\{a,b\})$.

Corollary 2.3. *If S is a right simple cyclic semigroup generated by a then for any $b \in S$, $Cay(S,\{a,b\})$ contain a Hamiltonian circuit.*

Proof. Suppose S is a right simple cyclic semigroup generated by a. From Theorem 2.1, there is a Hamiltonian circuit in $Cay(S,\{a\})$. Since $Cay(S,\{a\})$ is a spanning subgraph of $Cay(S,\{a,b\})$ for all $b \in S$ then $Cay(S,\{a,b\})$ for all $b \in S$ has a Hamiltonian circuit. $\square$

Restating Corollary 2.3 for any subset A of S, we have the following.

Corollary 2.4. *If S is a right simple cyclic semigroup generated by a then for any subset A of S which $a \in A$, $Cay(S,A)$ contain a Hamiltonian circuit.*

Theorem 2.5. *Let S be a semigroup with zero 0. If $S = \langle\{0, a\}\rangle$ for some $a \in S$ and $S \setminus \{0\}$ is a subsemigroup of S then $Cay(S, \{0, a\})$ contain a Hamiltonian path.*

Proof. Assume that $S \setminus \{0\}$ is a subsemigroup of S and there is $a \in S$ such that $S = \langle 0, a\rangle$. Since $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$, we have $S \setminus \{0\} = \langle a\rangle$. By Theorem 2.1, there is a Hamiltonian path in $Cay(S \setminus \{0\}, \{a\})$. But for all $y \in S$, $y \cdot 0 = 0$ so vertex y is adjacent to vertex 0 for all $y \in S$. Therefore $Cay(S, \{0, a\})$ contain a Hamiltonian path. $\square$

Theorem 2.6. *Let S be a semigroup with zero 0. Then for any subset A of S which $0 \in A$, $Cay(S, A)$ not contain a Hamiltonian circuit.*

Proof. Let A be a subset of S containing 0. Since $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$, we have that for all $y \in S \setminus \{0\}$, vertex y is adjacent to vertex 0 and vertex 0 is not adjacent to vertex y in $Cay(S, A)$. This implies that $Cay(S, A)$ not contain a Hamiltonian circuit. $\square$

Theorem 2.7. *Let S be a semigroup with identity e. If $S = \langle\{e, a\}\rangle$ for some $a \in S$ then $Cay(S, \{e, a\})$ contain a Hamiltonian path.*

Proof. Assume that S has order k and there is $a \in S$ such that $S = \langle e, a\rangle$. Since $x \cdot e = e \cdot x = x$ for all $x \in S$, $S \setminus \{e\} = \langle a\rangle$. By the proof of Theorem 2.1, $a, a^2, a^3, \ldots, a^k$ is a Hamiltonian path in $Cay(S \setminus \{e\}, \{a\})$. But $e \cdot a = a$ hence vertex e is adjacent to vertex a in $Cay(S, \{e, a\})$ so $e, a, a^2, a^3, \ldots, a^k$ is a Hamiltonian path in $Cay(S, \{e, a\})$. $\square$

Theorem 2.8. *Let S be a semigroup with identity e and $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$. Then for any nonempty subset A of S, $Cay(S, A)$ not contain a Hamiltonian circuit.*

Proof. Let A be a nonempty subset of S. Since $x \cdot e = e \cdot x = x$ for all $x \in S$, hence vertex e is adjacent to vertex y for all $y \in A$. Because $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$ so vertex y is not adjacent to vertex e for all $a \in S \setminus \{e\}$. This implies that $Cay(S, A)$ not contain a Hamiltonian circuit. $\square$

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