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# THE CONDITIONS OF SOME CAYLEY DIGRAPHS CONTAINING HAMILTONIAN PATH AND HAMILTONIAN CIRCUIT

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**Abstract**

For a finite semigroup $S$ and a nonempty subset $A$ of $S$ the *Cayley Digraph* of $S$ with respect to $A$, denoted by $Cay(S, A)$ is the directed graph with vertex set $S$ and arc set $\{(s, sa) \mid s \in S \text{ and } a \in A\}$. For digraph $D$, a directed path and a directed circuit which contain every vertex of $D$ is called a *Hamiltonian path* and *Hamiltonian circuit*, respectively. In this paper, we obtain some necessary and sufficient conditions of $S$ and $A$ such that $|A| \leq 2$ that $Cay(S, A)$ contain a Hamiltonian circuit and a Hamiltonian path.

## 1 Introduction

A *digraph* $D = (V, E)$ is defined by a set $V$ of vertices and a set $E$ of arcs. For vertices $u_1$ and $u_k$ in digraph $D$, a $u_1 - u_k$ *directed walk* in $D$ is an alternating sequence $u_1, e_1, u_2, ..., e_{k-1}, u_k$ of vertices and arcs, beginning with $u_1$ and ending with $u_k$, such that $e_i = (u_i, u_{i+1})$ for $i = 1, 2, ..., k-1$. If the vertices $u_1, u_2, ..., u_k$ are distinct, then the $u_1 - u_k$ directed walk is $u_1 - u_k$ *directed path*. If $u_1 = u_k$, where $k \geq 3$, and the vertices $u_1, u_2, ..., u_k$ are distinct, then the directed walk is called a *directed circuit*. A digraph $D$ is *strongly connected* if for every pair $u, v$ of vertices, $D$ contains both a $u - v$

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directed path and a $v-u$ directed path. A directed path and a directed circuit in $D$ are called a *Hamiltonian path* and *Hamiltonian circuit*, respectively if contain every vertex of $D$.

Hamiltonian paths and Hamiltonian circuits are the mathematical field of graph theory which are interested. The problem of finding the Hamiltonian path and the Hamiltonian circuit are NP-complete. They can apply to many problems, such as, pizza delivery, mail delivery, traveling sales man, garbage pick up, bus service/limosine service and reading gas meters. Furthermore, they have been applied to biology[1].

In 1878, Arthur Cayley introduced definition of the Cayley digraph of group. For $G$ is a finite group and $A$ is a set of generators of $G$. The Cayley digraph of $G$ with respect to $A$, denoted by $Cay(G, A)$ to be the directed graph with vertex set $G$ and arc set $\{(g, ga) : g \in G \text{ and } a \in A\}$. Moreover, Cayley studied properties of a group such as commutativity and the multiplication table can be recovered from the Cayley digraph. A well known conjecture of Lovasz states[14] that every Cayley digraph is a Hamiltonian path. There are many researchers that studied about this conjecture. In 1976, Nathanson[15] said that the finite group $G$ is generated by two element $a$ and $b$, such that $a^2 = b^3 = e$. If $|G| \geqslant 9|ab^2|$, then the Cayley digraph $Cay(G, \{a, b\})$ does not have a Hamiltonian path. In 1978, Holszyński and Strube[20] showed that every connected Cayley digraph on any abelian group has a Hamiltonian path. In 1982, Dragon[8] proved that if either $G$ is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian of odd order, then every connected Cayley digraph of $G$ is hamiltonian. In 1983, Marušič[5] showed that if either $G$ is a finite abelian group or a semidirect product of a cyclic group of prime order by a finite abelian group of odd order, then every connected Cayley digraph of $G$ is Hamiltonian circuit. In 2009, Pak and Radoičić[11] said that every finite group $G$ of size $|G| \geq 3$ has a generating set $A$ of size $|A| \leq log_2|G|$, such that the corresponding Cayley digraph $(G, A)$ contains a Hamiltonian circuit. In 2015, Suksumran and Panma proved Theorem 1.1, the digraph is strongly connected, this means the digraph may have a Hamiltonian path.

**Theorem 1.1.** *[19] $S$ is a right simple semigroup and $A \subseteq S$. A Cayley digraph $Cay(S, A)$ is strongly connected if and only if $S = \langle A \rangle$.*

In 2016, Khosravi [2] said that if $Cay(S, A)$ is strongly connected, then $S = \langle A \rangle$. Hence, we will consider some necessary and sufficient conditions of $S$ and $A$ that $Cay(S, A)$ has Hamiltonian paths or Hamiltonian circuits.

# 2 Main Results

First, we will show the conditions of $S$ and $A$ that $Cay(S, A)$ have a Hamiltonian paths or Hamiltonian circuits when $|A| = 1$.

**Theorem 2.1.** *$S$ is a cyclic semigroup generated by $a$ if and only if $Cay(S, \{a\})$ contain a Hamiltonian path.*

**Proof.** Let $S$ be a cyclic semigroup generated by $a$ with order $k$. Then $S = \{a, a^2, a^3, \ldots, a^k\}$ and $a^i \neq a^j$ for all $i, j \in \{1, 2, 3, \ldots, k\}$ which $i \neq j$. Since $a \cdot a = a^2$, vertex $a$ and vertex $a^2$ are adjacent in $Cay(S, \{a\})$. Since $a^2 \cdot a = a^3$, vertex $a^2$ and vertex $a^3$ are adjacent in $Cay(S, \{a\})$. Continue this process, we have a directed path from $a$ to $a^k$. That means $Cay(S, \{a\})$ contain a Hamiltonian path. Conversely, assume that there is $a \in S$ such that $Cay(S, \{a\})$ containing a Hamiltonian path say $H$. Suppose that $S \neq \langle a \rangle$. Hence there exits $a_0 \in \langle a \rangle$ and $b \in S \setminus \langle a \rangle$ such that vertex $a_0$ adjacent to vertex $b$ in directed path $H$. This implies $b = a_0 \cdot a \in \langle a \rangle$, contradiction. Therefore $S$ is a cyclic semigroup generated by $a$. □

**Theorem 2.2.** *$S$ is a right simple cyclic semigroup generated by $a$ if and only if $Cay(S, \{a\})$ contain a Hamiltonian circuit.*

**Proof.** Let $S$ be a right simple cyclic semigroup generated by $a$ with order $k$. By the proof of Theorem 2.1, we have a Hamiltonian path $a, a^2, a^3, \ldots, a^k$ in $Cay(S, \{a\})$. By Theorem 1.1, we have that $Cay(S, \{a\})$ is a strongly connected digraph. Hence the vertex $a^k$ must have a directed path to vertex $a$. This shows, $Cay(S, \{a\})$ contain a Hamiltonian circuit. Conversely, assume that there is $a \in S$ such that $Cay(S, \{a\})$ contain a Hamiltonian circuit length $k$. Then $Cay(S, \{a\})$ contain a Hamiltonian path length $k$. By Theorem 2.1, $S = \langle a \rangle = \{a, a^2, a^3, \ldots, a^k\}$ and $a^{k+1} = a$. Let $b \in S$. Hence $b = a^m$ for some $m \in \{1, 2, 3, \ldots, k\}$ so $b \cdot S = a^m \cdot \{a, a^2, a^3, \ldots, a^k\} = \{a^{m+1}, a^{m+2}, a^{m+3}, \ldots, a^{m+k}\} = S$. This shows that $b \cdot S = S$ for all $b \in S$. Therefore $S$ is a right simple cyclic semigroup generated by $a$. □

Notice that for any semigroup $S$ and $a \in S$ with $a \neq b$, it is easy to see that $Cay(S, \{a\})$ is a spanning subgraph of $Cay(S, \{a, b\})$. Hence a Hamiltonian path of $Cay(S, \{a\})$ is a Hamiltonian path of $Cay(S, \{a, b\})$ and a Hamiltonian circuit of $Cay(S, \{a\})$ is a Hamiltonian circuit of $Cay(S, \{a, b\})$.

**Corollary 2.3.** *If $S$ is a right simple cyclic semigroup generated by $a$ then for any $b \in S$, $Cay(S, \{a, b\})$ contain a Hamiltonian circuit.*

**Proof.** Suppose $S$ is a right simple cyclic semigroup generated by $a$. From Theorem 2.1, there is a Hamiltonian circuit in $Cay(S, \{a\})$. Since $Cay(S, \{a\})$ is a spanning subgraph of $Cay(S, \{a, b\})$ for all $b \in S$ then $Cay(S, \{a, b\})$ for all $b \in S$ has a Hamiltonian circuit. □

Restating Corollary 2.3 for any subset $A$ of $S$, we have the following.

**Corollary 2.4.** *If $S$ is a right simple cyclic semigroup generated by $a$ then for any subset $A$ of $S$ which $a \in A$, $Cay(S, A)$ contain a Hamiltonian circuit.*

**Theorem 2.5.** *Let $S$ be a semigroup with zero $0$. If $S = \langle \{0, a\} \rangle$ for some $a \in S$ and $S \setminus \{0\}$ is a subsemigroup of $S$ then $Cay(S, \{0, a\})$ contain a Hamiltonian path.*

**Proof.** Assume that $S \setminus \{0\}$ is a subsemigroup of $S$ and there is $a \in S$ such that $S = \langle 0, a \rangle$. Since $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$, we have $S \setminus \{0\} = \langle a \rangle$. By Theorem 2.1, there is a Hamiltonian path in $Cay(S \setminus \{0\}, \{a\})$. But for all $y \in S$, $y \cdot 0 = 0$ so vertex $y$ is adjacent to vertex $0$ for all $y \in S$. Therefore $Cay(S, \{0, a\})$ contain a Hamiltonian path. $\square$

**Theorem 2.6.** *Let $S$ be a semigroup with zero $0$. Then for any subset $A$ of $S$ which $0 \in A$, $Cay(S, A)$ not contain a Hamiltonian circuit.*

**Proof.** Let $A$ be a subset of $S$ containing $0$. Since $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$, we have that for all $y \in S \setminus \{0\}$, vertex $y$ is adjacent to vertex $0$ and vertex $0$ is not adjacent to vertex $y$ in $Cay(S, A)$. This implies that $Cay(S, A)$ not contain a Hamiltonian circuit. $\square$

**Theorem 2.7.** *Let $S$ be a semigroup with identity $e$. If $S = \langle \{e, a\} \rangle$ for some $a \in S$ then $Cay(S, \{e, a\})$ contain a Hamiltonian path.*

**Proof.** Assume that $S$ has order $k$ and there is $a \in S$ such that $S = \langle e, a \rangle$. Since $x \cdot e = e \cdot x = x$ for all $x \in S$, $S \setminus \{e\} = \langle a \rangle$. By the proof of Theorem 2.1, $a, a^2, a^3, \ldots, a^k$ is a Hamiltonian path in $Cay(S \setminus \{e\}, \{a\})$. But $e \cdot a = a$ hence vertex $e$ is adjacent to vertex $a$ in $Cay(S, \{e, a\})$ so $e, a, a^2, a^3, \ldots, a^k$ is a Hamiltonian path in $Cay(S, \{e, a\})$. $\square$

**Theorem 2.8.** *Let $S$ be a semigroup with identity $e$ and $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$. Then for any nonempty subset $A$ of $S$, $Cay(S, A)$ not contain a Hamiltonian circuit.*

**Proof.** Let $A$ be a nonempty subset of $S$. Since $x \cdot e = e \cdot x = x$ for all $x \in S$, hence vertex $e$ is adjacent to vertex $y$ for all $y \in A$. Because $a \cdot b \neq e$ for all $a, b \in S \setminus \{e\}$ so vertex $y$ is not adjacent to vertex $e$ for all $a \in S \setminus \{e\}$. This implies that $Cay(S, A)$ not contain a Hamiltonian circuit. $\square$

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