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ON AN IDENTITY INVOLVING INTEGER PARTITIONS SEQUENCES

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Abstract

In this short note we provide an q-series identity. By the use of the new concept of integer partition sequences, we give a bijective proof of this identity based in a famous Euler partition identity involving partitions into distinct parts and partitions in odd parts.

1 Introduction

In this paper we use the following standard notation:

$$(a,q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n,$$

for complex numbers a and q.

Our objective is to proof, using combinatorial arguments, the following q-series identity.

$$\sum_{n=1}^{\infty} q^{2n+1}(-q;q)_{2n}^{2n+1} = \sum_{n=1}^{\infty} q^{2n+1} \prod_{\substack{k \quad odd \\ k<2n-1}} \left(1+q^k+q^{2k}+\ldots+q^{k(1+2+2^2+\ldots+2^{f(2n+1,k)})}\right)^{2n+1}, \tag{1.1}$$

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where $f(2n+1,k)$ is the greatest power of two less than $\left[\frac{2n+1}{k}\right]$, and $[.]$ denotes the floor function.

For this purpose, we recall that a partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$, such that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. Each λ_i, $1 \leq i \leq k$, is called a part of the partition. For example, there are five partitions of 4: $4,\ 3+1,\ 2+2,\ 2+1+1$and $1+1+1+1$. Denote by $p(n)$ the number of integer partition of n, the generating function[1] the sequence $(p(n))_{n\geq 0}$ is given by the following series, for $|q| < 1$.

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

In this paper we deal with enumerative questions involving some generalization of the class of integer partitions. These questions are important in some areas of mathematics and physics. In Maslov [7], the author state that the number of partitions of n into exactly k parts, $p(n,k)$, is equal to the Bose and Fermi distributions with logarithmic accuracy for n identifies with energy and k with the number of particles.

A Plane partition π of n is a left-justified array of positive integers $(\pi_{i,j})_{i,j\geq 1}$ such that $\pi_{i,j} \geq max\{\pi_{i,j+1}, \pi_{i+1,j}\}$ and $n = \sum_{i,j\geq 1} \pi_{i,j}$. In other words, a plane partition is an planar array in which taking an element a_{ij}, considering it as a pivot, it is a part in two partitions one in horizontal direction and other in vertical direction. For instance, the following arrange is a plane partition of 52.

$$\begin{array}{ccccc} 6 & 5 & 4 & 4 & 2 \\ 5 & 4 & 3 & & \\ 4 & 3 & 2 & & \\ 3 & 2 & 1 & & \\ 2 & 1 & & & \end{array}$$

For $|q| < 1$, the generating function for plane partitions[2], is given by the infinite product as follows.

$$M(q) = \lim_{n\to\infty} \frac{1}{(q;q)_n^n}$$

Here we deal with bijective proofs involving classes of sequences of integer partitions. The importance of this type of proof is due to the fact of making a bijective association between one class of partition to another, using simple invertible operation, without the need to establish a generating function for both. For example, the Euler identity establishes that the number of partition into distinct parts is equal to the number of integer partitions into odd parts.

[1] Considerations of this generating function and several identities involving $(p(n))_{n\geq 0}$ can be find in Andrews [1, 2, 3] and more generally concepts of generating functions in Wilf [9].

[2] More information for this generating function can be find in Andrews [4].

For the proof of this fact let consider a partition into distinct parts λ. For each even part of λ, split it two parts until it became in a sum of odd parts. This operation is clearly invertible, and it gives a bijective proof of an important Euler partition identity. For example, the partition in distinct parts, $\lambda = 9+7+6+5+4+3+2$ is uniquely associated to the partition into odd parts $\mu = 9+7+5+3+3+3+1+1+1+1+1+1$. Bijective proofs like this and others more sophisticated can be find in Bressoud [5, 6] and Pak [8]. Considering generating functions, using simple cancellation laws, we can obtain the Euler identity mentioned here by the following equation.

$$\frac{1}{(q^2;q)_\infty} = (-q;q)_\infty$$

Definition 1. *A (n,k)-partition sequence is a finite sequence of partitions as $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ whose the sum of all parts is n in which the greatest part is k with at most k coordinates and k is a part of λ_1.*

Example 1. *For $n = 56$ and $k = 6$, the sequence $(6+5+4+3+2+1, 6+4+2+1, 6+2+1, 2+1, 5+3+2)$ is $(56,5)$-partition sequence.*

Analogous to the plane partitions we can represent a (n,k)-partition sequence as plane array, and obviously, a plane partition is a partition sequence. The representation of the $(56,6)$-partition sequence $(6+5+4+3+2+1, 6+4+2+1, 6+2+1, 2+1, 5+3+2)$ is given by the next diagram.

$$\begin{array}{cccccc} 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 4 & 2 & 1 & & \\ 6 & 2 & 1 & & & \\ 2 & 1 & & & & \\ 5 & 3 & 2 & & & \end{array}$$

The following diagrams represents all the twenty-five $(7,4)$-partition sequences. The bullet ($\bullet$) represents the empty partition.

$$\begin{array}{ccc} 4 & 1 & 1 \end{array}, \quad \begin{array}{ccc} 4 & 1 & 1 \\ 1 & & \end{array}, \quad \begin{array}{ccc} 4 & 1 & 1 \\ \bullet & & \\ 1 & & \end{array}, \quad \begin{array}{cc} 4 & 1 \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 4 & 1 \\ 1 & \\ 1 & \\ 1 & \end{array}, \quad \begin{array}{ccc} 4 & 1 & 1 \\ \bullet & & \\ \bullet & & \\ 1 & & \end{array}, \quad \begin{array}{cc} 4 & 1 \\ 1 & \\ \bullet & \\ 1 & \end{array},$$

$$\begin{array}{cc} 4 & 1 \\ \bullet & \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 4 & 1 \\ \bullet & \\ \bullet & \\ 1 & 1 \end{array}, \quad \begin{array}{c} 4 \\ 1 \\ 1 \\ 1 \end{array}, \quad \begin{array}{ccc} 4 & 2 & 1 \end{array}, \quad \begin{array}{cc} 4 & 2 \\ 1 & \end{array}, \quad \begin{array}{cc} 4 & 2 \\ \bullet & \\ 1 & \end{array}, \quad \begin{array}{cc} 4 & 2 \\ \bullet & \\ \bullet & \\ 1 & \end{array}, \quad \begin{array}{c} 4 \\ 2 \\ 1 \end{array}, \quad \begin{array}{c} 4 \\ 2 \\ \bullet \\ 1 \end{array},$$

$$\begin{array}{cc} 4 & 1 \\ \bullet & \\ 1 & \\ 1 & \end{array}, \quad \begin{array}{c} 4 \\ \bullet \\ 2 \\ 1 \end{array}, \quad \begin{array}{cc} 4 & \\ 2 & 1 \end{array}, \quad \begin{array}{cc} 4 & \\ \bullet & \\ 2 & 1 \end{array}, \quad \begin{array}{cc} 4 & \\ \bullet & \\ \bullet & \\ 2 & 1 \end{array}, \quad \begin{array}{cc} 4 & 3 \end{array}, \quad \begin{array}{c} 4 \\ 3 \end{array}, \quad \begin{array}{c} 4 \\ \bullet \\ 3 \end{array}, \quad \begin{array}{c} 4 \\ \bullet \\ \bullet \\ 3 \end{array}.$$

Concerning the number of the (n,k)-partition sequences we will provide the generating function for the sequence $(sp(n))_{n\geq 0}$, where $sp(n)$ is the function that enumerate the number of partition sequences of n. For this purpose, notice that the term

$$\frac{q^n}{(q;q)_n},$$

generates the first row of the partition sequence with the greatest part n while the term

$$\frac{1}{(q;q)_n^{n-1}},$$

generates the last l rows, $1 \leq l \leq n-1$. So we can establish the next proposition.

Proposition 1. *For* $|q| < 1$,

$$\sum_{n=0}^{\infty} sp(n)q^n = \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n^n}.$$

2 Main Result

In order to proof the veracity of the equation (1.1), we shall consider (n,k)-partition sequences in which in each row the parts are distinct and the greatest part k is odd. For example, the next sequence is a $(67,7)$-partition sequence of this kind.

$$\begin{array}{ccccccc}
7 & 6 & 5 & 4 & 3 & 2 & 1 \\
6 & 5 & 4 & & & & \\
5 & 3 & & & & & \\
6 & 4 & 2 & 1 & & & \\
3 & & & & & &
\end{array}$$

Using the bijective association row by row as described in the introduction of this paper, the partition sequence can be bijectively associated to the following partition sequence which all parts are odd.

$$\begin{array}{cccccccccccc}
7 & 5 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 3 & 3 & 1 & 1 & 1 & 1 & & & & & \\
5 & 3 & & & & & & & & & & \\
3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
3 & & & & & & & & & & &
\end{array}$$

From a partition sequence into distinct parts row by row, an even part is split in two parts, and if this parts are even, we should continue the spliting process. For example, the part 24 is associated to $3+3+3+3+3+3+3+3$. For the bijective association of an even part with sum of equal odd parts, we introduce the function $f(n,m)$ which is defined by the exponent of the greatest power of two lower than $\left[\frac{n}{m}\right]$, for an odd number n. The number of parts of an even parts splitted in sum of odd parts can be enumerated by the use of the function f. Indeed, in the Euler bijective proof for partitions into distinct parts, given an even part, say m, we have at most $1+2+2^2+\ldots+2^{f(2n+1,m)}$ equal odd parts j in which the sum of this parts are equal to m. Thus we can establish that the number of (n,k)-partitions sequences row by row into distinct parts, for k an odd number, is equal to the number of (n,k)-partitions sequences into odd parts in which a part j can appear repeated at most $(1+2+2^2+\ldots+2^{f(2n+1,j)})$ times without repetition of the part k.

Partitions with greatest part $2n+1$ into odd parts are generated by

$$q^{2n+1}\prod_{\substack{k \quad odd\\ k<2n-1}}(1+q^k+q^{2k}+\ldots+q^{(1+2+2^2+\ldots+2^{f(2n+1,k)})})$$

in each row. Using q-series, we have the following theorem.

Theorem 1. *For $|q|<1$, we have*

$$\begin{aligned}&\sum_{n=1}^{\infty} q^{2n+1}(-q;q)_{2n}^{2n+1}\\ &=\sum_{n=1}^{\infty} q^{2n+1}\prod_{\substack{k \quad odd\\ k<2n-1}}\left(1+q^k+q^{2k}+q^{3k}+\ldots+q^{k(1+2+2^2+\ldots+2^{f(2n+1,k)})}\right)^{2n+1}.\end{aligned}$$

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