

A NEW PROOF OF BAER-DEDEKIND THEOREM

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Abstract

In this paper we will give a new proof of the Baer-Dedekind theorem which classifies the groups in which each subgroup is normal, using the fact that these are torsion nilpotent groups whose class is less than or equal to 2.

1 Introduction

A subgroup H of a group G is called subnormal subgroup of G if there is an ascending chain of normal subgroups

$$H = G_m \trianglelefteq G_{m-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G \quad (\star)$$

of length $m \geq 0$. Considering all these chains of subgroups there is at least one of smaller length d and represented by $H = G_d \triangleleft G_{d-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$. The length of this shortest series is called the subnormal index or defect of H in G . Given a chain (\star) between H and G , we say that H is subnormal of defect less

Key words: Dedekind groups, nilpotency, quaternion group, torsion group.
2010 AMS Classification: Primary 20E34; Secondary 20F18.

than or equal to m . The **normal subgroups** are those with a defect ≤ 1 . If a group is nilpotent of class less than or equal to a certain $c \in \mathbb{N}$, then each of its subgroups is subnormal of defect $\leq c$ (e.g, p. 154 [9]). A natural question is about the validity of the reciprocal of the preceding statement. The first step in this direction was given by R. Dedekind (e.g, [5]) which determined all finite groups in which each of its subgroups are normal. Later in 1933, R. Baer in [3] extended this result to the arbitrary (infinite) groups. This fact is known as Baer-Dedekind's Theorem:

Main Theorem *Let G be a group. All subgroups of G are normal if and only if G is abelian or is a direct product of a quaternion group of order 8, an elementary abelian 2-group and a torsion abelian group whose elements have all odd order.*

The purpose of this article is to present a constructive proof of the above theorem, based mainly on the nilpotency and torsion properties of the groups in question. The original proofs can be found for example in [11, 15, 19].

Some examples, concepts and results of the theory of groups found in this article can be seen in [15, 17, 19]. In case of results on the theory of numbers (e.g, [8]). For Dedekind groups (e.g, [3, 11, 19]).

Let \mathbb{P} the set of prime numbers and G a group. Dr or \times will be the notation for the direct product of groups. If $x \in G$, denote by $o(x) = |x|$ the order of x . For all $h, g \in G$ define the conjugated of h by g as $h^g = g^{-1}hg$, the commutator of h and g as $[h, g] = h^{-1}g^{-1}hg = h^{-1}h^g$, more generally if $x_1, x_2, \dots, x_{n-1}, x_n \in G$ a simple commutator of weight $n \geq 2$ is defined recursively by rule $[x_1, x_2, \dots, x_{n-1}, x_n] = [[x_1, x_2, \dots, x_{n-1}], x_n]$. Let $\Phi(G)$ be the Frattini subgroup of G (for more details see p. 122-124 in [17]).

A torsion group (or periodic group) is a group all of whose elements have finite order. If the orders of the elements of a group are finite and limited, the group will be called the finite exponent group. Let $p \in \mathbb{P}$, we define a p -group as being a group in which each element has power order of p . A finite group is a p -group if and only if its order is a power of p . For an example of an infinite abelian p -group, see Prüfer group C_{p^∞} (e.g, p. 24 [15]). An elementary abelian p -group is an abelian group in which every nontrivial element has order p . Every elementary abelian p -group is a vector space over the prime field with p elements, and reciprocally (e.g, [6]). The torsion set $tor(G)$ of an group G is the subset of G consisting of all elements that have finite order. If G is a nilpotent group then $tor(G)$ is a fully-invariant subgroup such that $\frac{G}{tor(G)}$ is torsion-free (5.2.7 in [15]).

2 Preliminary.

Definition 2.1 A group is said *Dedekindian* if each of its subgroups is normal. A non-abelian Dedekindian group will be called *Hamiltonian*.

Remark i) Every abelian group is clearly Dedekindian. ii) If every cyclic subgroup of a group G is normal then G is Dedekindian. Indeed, let $\{1\} \neq H \not\leq G$. Put $1 \neq x \in H$ and $y \in G \setminus H$, as $\langle x \rangle \trianglelefteq G$ have $x^y \in \langle x \rangle \leq H$.

ii) The generalized quaternion group $Q_{2^n}, n \geq 3$, is a non-abelian finite 2-group having the following presentation $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} \rangle$. If $n = 3$ such a group has order 8 and is known as Hamilton's quaternions, the usual notation is $Q_8 = \langle i, j \mid i^4 = j^4 = 1, i^j = i^{-1} \rangle = \{\pm 1, \pm i, \pm j, \pm k\}$. If $\{1\} \neq L \leq Q_8$ we have $[Q_8, Q_8] \leq L$ therefore $L \trianglelefteq Q_8$ which implies that Q_8 is a **Hamiltonian group**. Q_{2^n} is not a Dedekindian group if $n > 3$, since the subgroup $\langle y \rangle$ is not normal in Q_{2^n} .

Counter-example The Dieder group of order 8 : $D_8 = \langle r, s \mid r^4 = s^2 = 1, r^s = r^{-1} \rangle$ is not Dedekindian, since the subgroup $\langle s \rangle$ is not normal, because $s^r = r^2 s \notin \langle s \rangle$.

The result below is well known, however we will briefly demonstrate for the convenience of the reader (see, e.g, p. 404 in [16]).

Lemma 2.2 *Let G be a Dedekindian group. Then G is a nilpotent group whose class is at most two.*

Proof. Let $x \in G$, have $\langle x \rangle \trianglelefteq G$, thus $N_G(\langle x \rangle) = G$. By [15, 1.6.13] have $\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \hookrightarrow \text{Aut}(\langle x \rangle)$. Note that $\text{Aut}(\langle x \rangle)$ is abelian (e.g, 1.5.5 [15]), thus

$$G' \leq C_G(\langle x \rangle), \forall x \in G \implies G' \leq \bigcap_{x \in G} C_G(\langle x \rangle) = \bigcap_{x \in G} C_G(x) = Z(G).$$

Therefore $\gamma_3(G) = \{1\}$ and so the result follows. \square

Remark The class of the Dedekindian Groups is closed for the formation of subgroups and quotients (the second follows from the Correspondence Theorem (e. g, Lemma 2.7.5 in [10]). However $Q_8 \times Q_8$ is not Dedekindian since the diagonal subgroup $D = \{(g, g) \mid g \in Q_8\}$ is not normal.

An more general example of a Hamiltonian group is described below.

Lemma 2.3 *Let $G = Q_8 \times E \times A$ a direct product of groups where E is elementary abelian 2-group, A an abelian torsion group such that all its elements have odd order. Then G is a Hamiltonian group.*

Proof. Clearly G is not an abelian group, because $Q_8 \leq G$. It is sufficient to prove that each cyclic subgroup of G is normal. Let $g = xyz, g_1 = x_1 y_1 z_1 \in G$,

where $x, x_1 \in Q_8$; $y, y_1 \in E$ and $z, z_1 \in A$. Have

$$g^{g_1} = (xyz)^{x_1 y_1 z_1} = x^{x_1} y^{y_1} z^{z_1} = x^{x_1} yz.$$

If $x^{x_1} = x$ then $g^{g_1} = xyz = g \in \langle g \rangle$, thus $\langle g \rangle^{g_1} = \langle g^{g_1} \rangle = \langle g \rangle \trianglelefteq G$.

Now suppose that $x^{x_1} = x^{-1}$ then $g^{g_1} = x^{-1}yz$. Note that $\text{mdc}(|z|, 4) = 1$ because $z \in A$. Now by Chinese Remainder Theorem (e.g, Theorem 7.2 in [8]) exist $n \in \mathbb{N}$ such that $n \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{|z|}$, which implies $n = 4k + 3 = s \cdot |z| + 1$ for certain $k, s \in \mathbb{Z}$. As $y \in E$ have $|y| \leq 2$, thus

$$g^{g_1} = x^{-1}yz = x^3 yz = x^{4k+3} y^{4k+3} z^{s \cdot |z| + 1} = x^n y^n z^n = (xyz)^n = g^n \in \langle g \rangle.$$

Therefore for all $g_1 \in G$ have $\langle g \rangle^{g_1} = \langle g^{g_1} \rangle = \langle g^n \rangle \leq \langle g \rangle$, thus $\langle g \rangle \trianglelefteq G$. \square

3 Reciprocal of Lemma 2.3.

Proposition 3.1 *Let G a Hamiltonian group. Then G is a torsion group, that is $\text{tor}(G) = G$.*

Proof. Let $x \in G$ such that $|x| = \infty$. If $x \notin Z(G)$, then exist $y \in G$ with $xy \neq yx$. As $\langle x \rangle \trianglelefteq G$ and $\langle y \rangle \trianglelefteq G$ follow that $[x, y] \in \langle x \rangle \cap \langle y \rangle$, so $\langle x \rangle \cap \langle y \rangle \neq 1$, that is, $\langle x \rangle \cap \langle y \rangle = \langle x^n \rangle$ for some $n \in \mathbb{N}$. Thus

$$x^n \in \langle y \rangle \implies [x^n, y] = 1 \implies (x^n)^y = x^n.$$

On the other hand $x^y = x^{-1}$ and so $y^{-1}x^ny = (x^n)^y = x^{-n} \neq x^n$ because $n \neq 0$. Therefore $x \in Z(G)$. Now by hypothesis $\{1\} \neq Z(G) \neq G$, so put $a \in G \setminus Z(G)$, then a has finite order and $xa \in G \setminus Z(G)$. Define $k = |xa| \cdot |a| < \infty$, then $x^k = x^k a^k = (xa)^k = 1$, a contradiction because x has infinite order, therefore G is a torsion group. \square

Proposition 3.2 *Let p be an odd prime and P a Dedekindian p -group. Then P is Abelian.*

Proof. Suppose by contradiction that P is a Dedekindian p -group and not Abelian. Then there are $a, b \in P$ with $ab \neq ba$. Consider the subgroup $L = \langle a, b \rangle = \langle a \rangle \cdot \langle b \rangle$. Since $|a|$ and $|b|$ are powers of the prime p have that L is a finite p -subgroup, Dedekindian and nonabelian of P . Therefore to prove such a proposition it is enough to suppose that P is finite. Put P a finite p -group, Dedekindian, nonabelian whose order is the smallest possible. By Burnside Basis Theorem (see, e. g, Theorem 5.50 in [17]) have $\frac{P}{\Phi(P)}$ is an elementary abelian p -group such that $|P/\Phi(P)| = p^2$, moreover $P' \leq \Phi(P)$. Note that if S is proper subgroup of P then S is abelian, moreover all quotient P/N with $1 \neq N \leq P$ also is abelian. If $N_1, N_2 \leq P$ such that $|N_1| = |N_2| = p$ and

$N_1 \neq N_2$, have that P/N_1 and P/N_2 are abelian groups. Then by Theorem 2.23 in [17] $P' \leq N_1 \cap N_2 = \{1\}$ so P is abelian, a contradiction. Therefore P has a unique subgroup N of order p such that $N = P'$. Define $\varphi : P \rightarrow P$ such that $x \mapsto x^p$. By Lemma 2.2 P is a nilpotent group whose class ≤ 2 , follows from Lemma 5.42 in [17] that $(xy)^p = [y, x]^{\frac{p(p-1)}{2}} x^p y^p$. As p is a odd prime have $p \mid \frac{p(p-1)}{2}$ and since P' has order p , follow that $[y, x]^{\frac{p(p-1)}{2}} = 1$. Thus φ is an endomorphism of P since

$$\varphi(xy) = (xy)^p = x^p y^p = \varphi(x) \cdot \varphi(y).$$

Clearly $Im(\varphi) \leq \Phi(P)$ because $\frac{P}{\Phi(P)}$ has exponent p . Since $\frac{P}{Ker(\varphi)} \cong Im(\varphi)$ have $|Ker(\varphi)| = \frac{|P|}{|Im(\varphi)|} \geq p^2$. Therefore there is more that a subgroup of order p in $Ker(\varphi) \leq P$ since it has exponent p , a contradiction by the first part of the proof of this Proposition. \square

Proposition 3.3 *Let Q be a finite Dedekindian 2-group such that Q is non-abelian minimal, that is, Q is not abelian and its own subgroups are abelian. Then $Q \cong Q_8$.*

Proof. Since subgroups of order p and p^2 are abelian (e.g, 1.6.15 [15]) have $|Q| = 2^n \geq 8$. Let $x, y \in Q$ with $xy \neq yx$. Consider the subgroup $W = \langle x, y \rangle = \langle x \rangle \cdot \langle y \rangle$ of Q . As W is a non-abelian finite Dedekindian 2-group it follows from the minimality of Q that $Q = W$, moreover $|\frac{Q}{\Phi(Q)}| = 4$ and Q has exactly three maximal subgroups, which are subgroups of index 2. By minimality of Q these subgroups are abelian, in addition we have $1 \not\leq Q' \leq \Phi(Q)$. Let's do the rest of this proof in two cases namely.

Case 1. Suppose there is a non-abelian quotient $\frac{Q}{N} = \langle \bar{x}, \bar{y} \rangle$ with $|N| = 2$, where $\bar{x} = x + N$ and $\bar{y} = y + N$ are the classes of x and y modulo N . In this case Q/N is a non-abelian minimal finite Dedekindian 2-group. The family of groups satisfying the hypotheses of the theorem is not empty since Q_8 belongs to this collection. By induction, suppose that the proposition is true for all groups whose order is less than that of Q . Then $\frac{Q}{N} \cong Q_8$ and since $|N| = 2$ have $|Q| = 16$. As $Q = \langle x \rangle \cdot \langle y \rangle$ and $\langle x \rangle, \langle y \rangle \trianglelefteq Q$ it can not occur that $L = \langle x \rangle \cap \langle y \rangle \neq \{1\}$ because otherwise Q would be the direct product of $\langle x \rangle \times \langle y \rangle$ so Q would be an abelian group, contradicting the hypothesis. Clearly $1 \neq |x| \neq 16$ and $1 \neq |y| \neq 16$ because otherwise Q would cyclic (abelian), contradicting the hypothesis. By Index Theorem (e.g, 1.3.11 [15]) $|\langle x \rangle \cdot \langle y \rangle| = \frac{|\langle x \rangle| \cdot |\langle y \rangle|}{|\langle x \rangle \cap \langle y \rangle|} = 16$ which implies that $|L| = 2$ or 4 otherwise one of the x or y should have order ≥ 16 , which is impossible. In any case, in Q there must be an element of order 8, without loss of generality suppose that x has such order. Clearly $N \leq \langle x \rangle$ because otherwise $Q = \langle x \rangle \times N$ which implies that Q would be Abelian, an absurd. Therefore as $|N| = 2$ have $N = \langle x^4 \rangle$. If $|L| = 2$ then $|y| = 4$, and since $L \leq \langle x \rangle$ follows that $x^4 = y^2$, where \bar{y} would have order 2, but this is impossible since the generators of $\frac{Q}{N} \cong Q_8$ have to have order 4. So

we have $|x| = |y| = 8$. Then x inverts y modulo N , that is $\bar{x}^y = \bar{x}^{-1}$. Then in Q we have $x^y = x^{-1} = x^7$ or $x^y = x^{-1}x^4 = x^3$. Since $J = \langle x^2 \rangle \langle y \rangle < Q$ it follows that J is abelian so $(x^2)^y = x^2$. Therefore $x^2 = (x^2)^y = (x^y)^2 = x^{14} = x^6$ a contradiction because the order of x is equal to 8. Thus, case 1 can not occur.

Caso 2. By the negative of case 1 together with the third isomorphism theorem, consider from now on that all proper quotients of Q are abelian. With an argument analogous to that used in Proposition 3.2 we have that Q' is the only subgroup of order 2 in Q . By Theorem 5.46 in [17] we have that the group Q must be cyclic or generalized quaternion, but since Q is Dedekind and noncyclic we have $Q \cong Q_8$. \square

Lemma 3.4 *Let D be a non-abelian Dedekind 2-group. Then there exists a subgroup B of D isomorphic to Q_8 , and in addition there is a maximal subgroup E of D with the property that $E \cap B = \{1\}$.*

Proof. Since D is not abelian there are a, b in D such that $ab \neq ba$. By Index Theorem (e.g. 1.3.11 [15]) $W = \langle a, b \rangle = \langle a \rangle \cdot \langle b \rangle \leq D$ is a Hamiltonian finite 2-subgroup. If W is minimal non-abelian we have by Proposition 3.3 that $W = W_0 = B \cong Q_8$ and the Lemma will be proved. Otherwise, there exists a non-abelian maximal subgroup W_1 of W , whose index is 2 (e.g. Theorem 5.40 [17]). If W_1 is minimal non-abelian we have by Proposition 3.3 that $W_1 = B \cong Q_8$ and the Lemma will be proved. This must be repeated if necessary until we find a subgroup W_j of D ($j \geq 0$) whose all of its maximal subgroups have order 4 (these are all abelian), since in this case W_j will be non-abelian minimal, and the proof will be completed by Proposition 3.3. Finally, let $\mathcal{L} = \{X \leq D \mid X \cap B = \{1\}\}$. Clearly $\{1\} \in \mathcal{L} \neq \emptyset$. By the **Lemma of Zorn** there exists a maximal subgroup E of D with the following property $E \cap B = \{1\}$. \square

Lemma 3.5 *Let D be a Dedekindian 2-group containing Q_8 and consider $H = \langle a \rangle$ a subgroup of order 4 in D . Then $H \cap Q_8 \neq \{1\}$ and furthermore $a^2 = -1$.*

Proof. Suppose that $H \cap Q_8 = \{1\}$. As D is Dedekind we have $Q_8 \cdot H = Q_8 \times H$. Now note that $\langle ia \rangle = \{1, ia, -a^2, -ia^3\}$ and moreover $(ia)^j = -ia \in \langle ia \rangle$ since that $\langle ia \rangle \leq D$, a contradiction, therefore $H \cap Q_8 \neq \{1\}$ thus $|H \cap Q_8| = 2$ or 4. If $|H \cap Q_8| = 2$ is trivial. On the other hand, if $|H \cap Q_8| = 4$ then $H \cap Q_8 = H = \langle \pm\gamma \rangle$ for some $\gamma = i, j$ or k , where it follows that $a^2 = (\pm\gamma)^2 = -1$. \square

Corollary 3.6 *Let D be a Dedekindian 2-group containing Q_8 and consider E a non-trivial subgroup of D such that $E \cap Q_8 = \{1\}$. Then E has exponent 2 (E is an elementary abelian 2-group).*

Proof. Let $a \neq 1 \in E$ with $|a| = 2^k$, where $k \geq 2$. Clearly $b = a^{2^{k-2}} \in E$ and $|b| = 4$. As $E \cap Q_8 = \{1\}$ have $\langle b \rangle \cap Q_8 = \{1\}$ which contradicts Lemma 3.5, thus $|a| = 2$. \square

Proposition 3.7 *Let D be a Hamiltonian 2-group. Then $D = Q_8 \times E$, where E is trivial or an elementary abelian 2-group.*

Proof. By Lemma 3.4 we can consider that Q_8 is a subgroup of D and in addition there is a maximal subgroup E of D with the property $E \cap Q_8 = \{1\}$, thus $Q_8 \cdot E = Q_8 \times E \leq D$. To prove the proposition it is enough to prove that if a belongs to D then a also belongs to $Q_8 \times E$.

Let $a \in D$ whose order is equal to 2. If $a \in Q_8$ or $a \in E$ it is trivial. If $E = \{1\}$ follows from its maximality that $\langle a \rangle \cap Q_8 \neq \{1\}$ thus $a \in Q_8 \times E$. Now suppose that $E \neq \{1\}$ and $a \in D \setminus (Q_8 \cup E)$. Consider $\langle a \rangle \cdot E = \langle a \rangle \times E$. Clearly $\langle a \rangle \times E \neq D$ because that $Q_8 \leq D$, thus by maximality of E exist $1 \neq b \in E$ such that $q = ab \in Q_8$. Therefore $a = qb^{-1} \in Q_8 \times E$.

Consider now $a \in D \setminus (Q_8 \cup E)$ such that $|a| = 4$. Again if $a \in Q_8$, it is trivial. By Lemma 3.5 have $a^2 = -1 \in Q_8$, moreover as D is Dedekind have that $\langle a \rangle \trianglelefteq D$ so if $s \in \{i, j, k\}$ have $a^s = a$ or $a^s = a^3$. Note that if $a^i = a^3$ and $a^j = a^3$ then $a^k = a^9 = a$. Testing all possibilities we noticed that one of the following equations at least is true: $a^i = a$ or $a^j = a$ or $a^k = a$. Assume without loss of generality that $a^i = a$ then $(ia)^2 = i^2 a^2 = 1$ so $|ia| = 2$. Then as we saw in the first part of this proof we have $ia \in Q_8 \times E$ and so $a \in Q_8 \times E$ too.

Let's now show that there is no element in group D whose order is greater than or equal to 8. Let $a \in D$ with $|a| = 2^k$, where $k > 3$, clearly $b = a^{2^{k-3}}$ satisfies $|b| = 8$. It is enough to show that such b do not exist in D . Put $a \in D$ with $|a| = 8$ then $\langle a^2 \rangle = \{id, a^2, a^4, a^6\}$. By Lemma 3.5 we have $a^4 = -1 \in Q_8$ and in addition for the second part of this proof we have $a^2 = s.m \in Q_8 \times E$. Clearly $s \neq \pm 1$ because that $|a^2| = 4$ a contradiction, so $s \in \{\pm i, \pm j, \pm k\}$. Suppose without loss of generality that $s = i$. Thus $(a^2)^j = -im = -a^2 = a^6$. As $\langle a \rangle \trianglelefteq D$ and $(a^2)^j = a^6$ we have $a^j \neq a$ and $a^j \neq a^5$. Therefore the only possibilities for a^j are $a^j = a^3$ or $a^j = a^7$. Suppose first that $a^j = a^3$ then $(ja)^2 = jaja = j^2 a^4 = 1$, now if $a^j = a^7$ then $(ja)^4 = jajajaja = j^2 a^8 jaja = j^3 aja = j^4 a^8 = 1$, thus $|ja| = 2$ or 4. Again for the same reasons as above $ja \in Q_8 \times E$ and so $a \in Q_8 \times E$. Then we prove that $D = Q_8 \times E$, in particular if E is trivial then $D = Q_8$. \square

Let us now use the results seen in this section to prove the following proposition. This is the reciprocal for the Baer-Dedekind Theorem.

Proposition 3.8 *Let G be a Hamiltonian group, then $G \cong Q_8 \times E \times A$, where A is an abelian torsion group whose elements have odd order and E is an elementary abelian 2-group.*

Proof. By Lemma 2.2 and Proposition 3.1 have that G is a nilpotent torsion group whose class is at most two, so $\text{tor}(G) = G$. By 5.2.7 in [15] have

$$G = Dr_{p \in \mathbb{P}} T_p = D \times Dr_{p \in \mathbb{P} \setminus \{2\}} T_p$$

where Dr denotes the direct product of groups, T_p is the unique maximum p -subgroup of G and \mathbb{P} is the set of all prime positive integers such that $D = T_2$. Now as T_p is a Dedekind group for each odd prime number p , follows from Proposition 3.2 that each T_p (odd p) is abelian which implies that $A = Dr_{p \in \mathbb{P} \setminus \{2\}} T_p$ is an abelian torsion group whose elements have odd order. Clearly D is not abelian otherwise G would be an abelian group. Then D is a Dedekindian non-abelian 2-group. By Proposition 3.7 have $D \cong Q_8 \times E$, where E is an elementary abelian 2-group. Therefore $G \cong Q_8 \times E \times A$. \square

We come to the main theorem whose original proof can be found in (for infinite groups, e.g. [3, 11, 19]), (for finite groups, e.g. [5]) and (for generalizations, e.g. [1, 2, 12, 18, 7, 14, 4, 13]).

Proof of the Main Theorem: It follows from Lemma 2.3 and Proposition 3.8. \square

References

- [1] I. N. Abramovskii, Locally Generalized Hamiltonian Groups, *Siberian Mathematical Journal* **7** (1966), 391-393.
- [2] I. N. Abramovskii, The Structure of Locally Generalized Hamiltonian Groups, *Leningrad Gos. Ped. Inst. Učen. Zap.* **302** (1967), 43-49.
- [3] R. Baer, Situation der Untergruppen und Struktur der Gruppe, *S. Berlin-Heidelberg. Akad. Wiss* **2** (1933), 12-17.
- [4] Cappitt, D. Generalized Dedekind groups. *Journal of Algebra.* (17): 310-316 (1971).
- [5] R. Dedekind, Über Gruppen, deren Sämtliche Teiler Normalteiler Sind, *Math. Ann.* **48** (1897), 548-561.
- [6] L. Fuchs, *Infinite Abelian Groups*, v. I. Academic Press, 1970.
- [7] X. Guo and J. Wang, *On generalized Dedekind groups*, *Acta Mathematica Hungarica.* **122** (2009), 37-44.
- [8] H. L. Keng, *Introduction to number theory*, Berlin-Heidelberg, 1982.
- [9] M. Hall, *The Theory of groups*, Macmillan, New York, 1959.
- [10] I. N. Herstein, *Topics in Algebra*, 2nd ed. Jonh Wiley & Sons, 1975.
- [11] J. C. Lennox and S. E. Stonehewer, *Subnormal Subgroups of Groups*, Oxford Mathematical Monographs, New York, 1987.
- [12] D. S. Nymann, Dedekind Groups, *Pacific J. Math.* **21** (1967), 153-159.
- [13] A.L.P. Porto and V. R. de Bessa, Profinite Dedekind groups, *Submitted*.
- [14] A. L. P. Porto, Profinite Cappitt Groups, *Quaestiones Mathematicae*, accepted for publication, 2020.
- [15] D. J. S. Robinson, *A Course in the Theory of Groups*, New York - Heidelberg - Berlin: Springer, 1980.
- [16] J. E. Roseblade, On Group in which every Subgroup is Subnormal, *J. Àlgebra* **2** (1965), 402-412.
- [17] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed. Springer-Verlag, 1995.
- [18] N. N. Vil'jams, Metadedekind and Metahamiltonian Groups, *Math. USSR-Sb.* **5** (1968), 599-616.
- [19] H. Zassenhaus, *The Theory of Groups*, 2nd ed. New York: Chelsea, 1958.