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A NEW PROOF OF BAER-DEDEKIND THEOREM

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Abstract

In this paper we will give a new proof of the Baer-Dedekind theorem which classifies the groups in which each subgroup is normal, using the fact that these are torsion nilpotent groups whose class is less than or equal to 2.

1 Introduction

A subgroup H of a group G is called subnormal subgroup of G if there is an ascending chain of normal subgroups

$$H = G_m \trianglelefteq G_{m-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G \quad (\star)$$

of length $m \geq 0$. Considering all these chains of subgroups there is at least one of smaller length d and represented by $H = G_d \triangleleft G_{d-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$. The length of this shortest series is called the subnormal index or defect of H in G. Given a chain $(\star)$ between H and G, we say that H is subnormal of defect less

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than or equal to m. The **normal subgroups** are those with a defect ≤ 1. If a group is nilpotent of class less than or equal to a certain $c \in \mathbb{N}$, then each of its subgroups is subnormal of defect $\leq c$ (e.g, p. 154 [9]). A natural question is about the validity of the reciprocal of the preceding statement. The first step in this direction was given by R. Dedekind (e.g, [5]) which determined all finite groups in which each of its subgroups are normal. Later in 1933, R. Baer in [3] extended this result to the arbitrary (infinite) groups. This fact is known as Baer-Dedekind's Theorem:

Main Theorem *Let G be a group. All subgroups of G are normal if and only if G is abelian or is a direct product of a quaternion group of order* 8, *an elementary abelian* 2*-group and a torsion abelian group whose elements have all odd order.*

The purpose of this article is to present a constructive proof of the above theorem, based mainly on the nilpotency and torsion properties of the groups in question. The original proofs can be found for example in [11, 15, 19].

Some examples, concepts and results of the theory of groups found in this article can be seen in [15, 17, 19]. In case of results on the theory of numbers (e.g, [8]). For Dedekind groups (e.g, [3, 11, 19]).

Let $\mathbb{P}$ the set of prime numbers and G a group. Dr or $\times$ will be the notation for the direct product of groups. If $x \in G$, denote by $o(x) = |x|$ the order of x. For all $h, g \in G$ define the conjugated of h by g as $h^g = g^{-1}hg$, the commutator of h and g as $[h, g] = h^{-1}g^{-1}hg = h^{-1}h^g$, more generally if $x_1, x_2, \ldots, x_{n-1}, x_n \in G$ a simple commutator of weight $n \geq 2$ is defined recursively by rule $[x_1, x_2, \ldots, x_{n-1}, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$. Let $\Phi(G)$ be the Frattini subgroup of G (for more details see p. 122-124 in [17]).

A torsion group (or periodic group) is a group all of whose elements have finite order. If the orders of the elements of a group are finite and limited, the group will be called the finite exponent group. Let $p \in \mathbb{P}$, we define a p-group as being a group in which each element has power order of p. A finite group is a p-group if and only if its order is a power of p. For an example of an infinite abelian p-group, see Prüfer group C_{p^∞} (e.g, p. 24 [15]). An elementary abelian p-group is an abelian group in which every nontrivial element has order p. Every elementary abelian p-group is a vector space over the prime field with p elements, and reciprocally (e.g, [6]). The torsion set $tor(G)$ of an group G is the subset of G consisting of all elements that have finite order. If G is a nilpotent group then $tor(G)$ is a fully-invariant subgroup such that $\frac{G}{tor(G)}$ is torsion-free (5.2.7 in [15]).

2 Preliminary.

Definition 2.1 A group is said *Dedekindian* if each of its subgroups is normal. A non-abelian Dedekindian group will be called *Hamiltonian*.

Remark i) Every abelian group is clearly Dedekindian. ii) If every cyclic subgroup of a group G is normal then G is Dedekindian. Indeed, let $\{1\} \neq H \gneqq G$. Put $1 \neq x \in H$ and $y \in G \backslash H$, as $\langle x \rangle \trianglelefteq G$ have $x^y \in \langle x \rangle \leqslant H$.

ii) The generalized quarternion group $Q_{2^n}, n \geq 3$, is a non-abelian finite 2-group having the following presentation $Q_{2^n} = \langle x, y \,|\, x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} \rangle$. If $n = 3$ such a group has order 8 and is known as Hamilton's quaternions, the usual notation is $Q_8 = \langle i, j \,|\, i^4 = j^4 = 1, i^j = i^{-1} \rangle = \{\pm 1, \pm i, \pm j, \pm k\}$. If $\{1\} \neq L \leqslant Q_8$ we have $[Q_8, Q_8] \leqslant L$ therefore $L \trianglelefteq Q_8$ which implies that Q_8 is a **Hamiltonian group**. Q_{2^n} is not a Dedekindian group if $n > 3$, since the subgroup $\langle y \rangle$ is not normal in Q_{2^n}.

Counter-example The Diedral group of order 8 : $D_8 = \langle r, s \,|\, r^4 = s^2 = 1, r^s = r^{-1} \rangle$ is not Dedekindian, since the subgroup $\langle s \rangle$ is not normal, because $s^r = r^2 s \notin \langle s \rangle$.

The result below is well known, however we will briefly demonstrate for the convenience of the reader (see, e.g, p. 404 in [16]).

Lemma 2.2 *Let G be a Dedekindian group. Then G is a nilpotent group whose class is at most two.*

Proof. Let $x \in G$, have $\langle x \rangle \trianglelefteq G$, thus $N_G(\langle x \rangle) = G$. By [15, 1.6.13] have $\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \hookrightarrow Aut(\langle x \rangle)$. Note that $Aut(\langle x \rangle)$ is abelian (e.g, 1.5.5 [15]), thus

$$G' \leqslant C_G(\langle x \rangle), \forall\, x \in G \Longrightarrow G' \leqslant \bigcap_{x \in G} C_G(\langle x \rangle) = \bigcap_{x \in G} C_G(x) = Z(G).$$

Therefore $\gamma_3(G) = \{1\}$ and so the result follows. □

Remark The class of the Dedekindian Groups is closed for the formation of subgroups and quotients (the second follows from the Correspondence Theorem (e. g, Lemma 2.7.5 in [10]). However $Q_8 \times Q_8$ is not Dedekindian since the diagonal subgroup $D = \{(g, g) | g \in Q_8\}$ is not normal.

An more general example of a Hamiltonian group is described below.

Lemma 2.3 *Let $G = Q_8 \times E \times A$ a direct product of groups where E is elementary abelian 2-group, A an abelian torsion group such that all its elements have odd order. Then G is a Hamiltonian group.*

Proof. Clearly G is not an abelian group, because $Q_8 \leqslant G$. It is sufficient to prove that each cyclic subgroup of G is normal. Let $g = xyz, g_1 = x_1 y_1 z_1 \in G$, where $x, x_1 \in Q_8$; $y, y_1 \in E$ and $z, z_1 \in A$. Have

$$g^{g_1} = (xyz)^{x_1 y_1 z_1} = x^{x_1} y^{y_1} z^{z_1} = x^{x_1} yz.$$

If $x^{x_1} = x$ then $g^{g_1} = xyz = g \in \langle g \rangle$, thus $\langle g \rangle^{g_1} = \langle g^{g_1} \rangle = \langle g \rangle \trianglelefteq G$.

Now suppose that $x^{x_1} = x^{-1}$ then $g^{g_1} = x^{-1}yz$. Note that $mdc(|z|, 4) = 1$ because $z \in A$. Now by Chinese Remainder Theorem (e.g, Theorem 7.2 in [8]) exist $n \in \mathbb{N}$ such that $n \equiv 3 (mod\, 4)$ and $n \equiv 1 (mod |z|)$, which implies $n = 4k + 3 = s \cdot |z| + 1$ for certain $k, s \in \mathbb{Z}$. As $y \in E$ have $|y| \leq 2$, thus

$$g^{g_1} = x^{-1}yz = x^3yz = x^{4k+3}y^{4k+3}z^{s\cdot|z|+1} = x^n y^n z^n = (xyz)^n = g^n \in \langle g \rangle.$$

Therefore for all $g_1 \in G$ have $\langle g \rangle^{g_1} = \langle g^{g_1} \rangle = \langle g^n \rangle \leqslant \langle g \rangle$, thus $\langle g \rangle \trianglelefteq G$.

□

3 Reciprocal of Lemma 2.3.

Proposition 3.1 *Let G a **Hamiltonian group**. Then G is a torsion group, that is $tor(G) = G$.*

Proof. Let $x \in G$ such that $|x| = \infty$. If $x \notin Z(G)$, then exist $y \in G$ with $xy \neq yx$. As $\langle x \rangle \trianglelefteq G$ and $\langle y \rangle \trianglelefteq G$ follow that $[x, y] \in \langle x \rangle \cap \langle y \rangle$, so $\langle x \rangle \cap \langle y \rangle \neq 1$, that is, $\langle x \rangle \cap \langle y \rangle = \langle x^n \rangle$ for some $n \in \mathbb{N}$. Thus

$$x^n \in \langle y \rangle \Longrightarrow [x^n, y] = 1 \Longrightarrow (x^n)^y = x^n.$$

On the other hand $x^y = x^{-1}$ and so $y^{-1}x^n y = (x^n)^y = x^{-n} \neq x^n$ because $n \neq 0$. Therefore $x \in Z(G)$. Now by hypothesis $\{1\} \neq Z(G) \neq G$, so put $a \in G \setminus Z(G)$, then a has finite order and $xa \in G \setminus Z(G)$. Define $k = |xa| \cdot |a| < \infty$, then $x^k = x^k a^k = (xa)^k = 1$, a contradiction because x has infinite order, therefore G is a torsion group. □

Proposition 3.2 *Let p be an odd prime and P a Dedekindian p-group. Then P is Abelian.*

Proof. Suppose by contradiction that P is a Dedekindian p-group and not Abelian. Then there are $a, b \in P$ with $ab \neq ba$. Consider the subgroup $L = \langle a, b \rangle = \langle a \rangle \cdot \langle b \rangle$. Since $|a|$ and $|b|$ are powers of the prime p have that L is a finite p-subgroup, Dedekindian and nonabelian of P. Therefore to prove such a proposition it is enough to suppose that P is finite. Put P a finite p-group, Dedekindian, nonabelian whose order is the smallest possible. By Burnside Basis Theorem (see, e. g, Theorem 5.50 in [17]) have $\frac{P}{\Phi(P)}$ is an elementary abelian p-group such that $|P/\Phi(P)| = p^2$, moreover $P' \leqslant \Phi(P)$. Note that if S is proper subgroup of P then S is abelian, moreover all quocient P/N with $1 \neq N \leqslant P$ also is abelian. If $N_1, N_2 \leqslant P$ such that $|N_1| = |N_2| = p$ and

$N_1 \neq N_2$, have that P/N_1 and P/N_2 are abelian groups. Then by Theorem 2.23 in [17] $P' \leqslant N_1 \cap N_2 = \{1\}$ so P is abelian, a contradiction. Therefore P has a unique subgroup N of order p such that $N = P'$. Define $\varphi : P \longrightarrow P$ such that $x \longmapsto x^p$. By Lemma 2.2 P is a nilpotent group whose class ≤ 2, follows from Lemma 5.42 in [17] that $(xy)^p = [y,x]^{\frac{p(p-1)}{2}} x^p y^p$. As p is a odd prime have $p|\frac{p(p-1)}{2}$ and since P' has order p, follow that $[y,x]^{\frac{p(p-1)}{2}} = 1$. Thus φ is a endomorphism of P since

$$\varphi(xy) = (xy)^p = x^p y^p = \varphi(x) \cdot \varphi(y).$$

Clearly $Im(\varphi) \leqslant \Phi(P)$ because $\frac{P}{\Phi(P)}$ has exponent p. Since $\frac{P}{Ker(\varphi)} \cong Im(\varphi)$ have $|Ker(\varphi)| = \frac{|P|}{|Im(\varphi)|} \geqslant p^2$. Therefore there is more that a subgroup of order p in $Ker(\varphi) \leqslant P$ since it has exponent p, a contradiction by the first part of the proof of this Proposition. □

Proposition 3.3 *Let Q be a finite Dedekindian 2-group such that Q is non-abelian minimal, that is, Q is not abelian and its own subgroups are abelian. Then $Q \cong Q_8$.*

Proof. Since subgroups of order p and p^2 are abelian (e.g, 1.6.15 [15]) have $|Q| = 2^n \geq 8$. Let $x, y \in Q$ with $xy \neq yx$. Consider the subgroup $W = \langle x, y\rangle = \langle x\rangle \cdot \langle y\rangle$ of Q. As W is a non-abelian finite Dedekindian 2-group it follows from the minimality of Q that $Q = W$, moreover $|\frac{Q}{\Phi(Q)}| = 4$ and Q has exactly three maximal subgroups, which are subgroups of index 2. By minimality of Q these subgroups are abelian, in addition we have $1 \lneqq Q' \leqslant \Phi(Q)$. Let's do the rest of this proof in two cases namely.

Case 1. Suppose there is a non-abelian quotient $\frac{Q}{N} = \langle \bar{x}, \bar{y}\rangle$ with $|N| = 2$, where $\bar{x} = x + N$ and $\bar{y} = y + N$ are the classes of x and y modulo N. In this case Q/N is a non-abelian minimal finite Dedekindian 2-group. The family of groups satisfying the hypotheses of the theorem is not empty since Q_8 belongs to this collection. By induction, suppose that the proposition is true for all groups whose order is less than that of Q. Then $\frac{Q}{N} \cong Q_8$ and since $|N| = 2$ have $|Q| = 16$. As $Q = \langle x\rangle \cdot \langle y\rangle$ and $\langle x\rangle, \langle y\rangle \trianglelefteq Q$ it can not occur that $L = \langle x\rangle \cap \langle y\rangle \neq \{1\}$ because otherwise Q would be the direct product of $\langle x\rangle \times \langle y\rangle$ so Q would be an abelian group, contradicting the hypothesis. Clearly $1 \neq |x| \neq 16$ and $1 \neq |y| \neq 16$ because otherwise Q would cyclyc (abelian), contradicting the hypothesis. By Index Theorem (e.g, 1.3.11 [15]) $|\langle x\rangle \cdot \langle y\rangle| = \frac{|\langle x\rangle| \cdot |\langle y\rangle|}{|\langle x\rangle \cap \langle y\rangle|} = 16$ which implies that $|L| = 2$ or 4 otherwise one of the x or y should have order ≥ 16, which is impossible. In any case, in Q there must be an element of order 8, without loss of generality suppose that x has such order. Clearly $N \leqslant \langle x\rangle$ because otherwise $Q = \langle x\rangle \times N$ which implies that Q would be Abelian, an absurd. Therefore as $|N| = 2$ have $N = \langle x^4\rangle$. If $|L| = 2$ then $|y| = 4$, and since $L \leqslant \langle x\rangle$ follows that $x^4 = y^2$, where $\bar{y}$ would have order 2, but this is impossible since the generators of $\frac{Q}{N} \cong Q_8$ have to have order 4. So we have $|x| = |y| = 8$. Then x inverts y modulo N, that is $\bar{x}^{\bar{y}} = \bar{x}^{-1}$. Then in Q we have $x^y = x^{-1} = x^7$ or $x^y = x^{-1}x^4 = x^3$. Since $J = \langle x^2 \rangle \langle y \rangle < Q$ it follows that J is abelian so $(x^2)^y = x^2$. Therefore $x^2 = (x^2)^y = (x^y)^2 = x^{14} = x^6$ a contradiction because the order of x is equal to 8. Thus, case 1 can not occur.

Caso 2. By the negative of case 1 together with the third isomorphism theorem, consider from now on that all proper quotients of Q are abelian. With an argument analogous to that used in Proposition 3.2 we have that Q' is the only subgroup of order 2 in Q. By Theorem 5.46 in [17] we have that the group Q must be cyclic or generalized quaternion, but since Q is Dedekind and noncyclic we have $Q \cong Q_8$. $\square$

Lemma 3.4 *Let D be a non-abelian Dedekind 2-group. Then there exists a subgroup B of D isomorphic to Q_8, and in addition there is a maximal subgroup E of D with the property that $E \cap B = \{1\}$.*

Proof. Since D is not abelian there are a, b in D such that $ab \neq ba$. By Index Theorem (e.g, 1.3.11 [15]) $W = \langle a, b \rangle = \langle a \rangle \cdot \langle b \rangle \leqslant D$ is a Hamiltonian finite 2-subgroup. If W is minimal non-abelian we have by Proposition 3.3 that $W = W_0 = B \cong Q_8$ and the Lemma will be proved. Otherwise, there exists a non-abelian maximal subgroup W_1 of W, whose index is 2 (e.g, Theorem 5.40 [17]). If W_1 is minimal non-abelian we have by Proposition 3.3 that $W_1 = B \cong Q_8$ and the Lemma will be proved. This must be repeated if necessary until we find a subgroup W_j of $D (j \geq 0)$ whose all of its maximal subgroups have order 4 (these are all abelian), since in this case W_j will be non-abelian minimal, and the proof will be completed by Proposition 3.3. Finally, let $\mathcal{L} = \{X \leqslant D \mid X \cap B = \{1\}\}$. Clearly $\{1\} \in \mathcal{L} \neq \emptyset$. By the **Lemma of Zorn** there exists a maximal subgroup E of D with the following property $E \cap B = \{1\}$. $\square$

Lemma 3.5 *Let D be a Dedekindian 2-group containing Q_8 and consider $H = \langle a \rangle$ a subgroup of order 4 in D. Then $H \cap Q_8 \neq \{1\}$ and furthermore $a^2 = -1$.*

Proof. Suppose that $H \cap Q_8 = \{1\}$. As D is Dedekind we have $Q_8 \cdot H = Q_8 \times H$. Now note that $\langle ia \rangle = \{1, ia, -a^2, -ia^3\}$ and moreover $(ia)^j = -ia \in \langle ia \rangle$ since that $\langle ia \rangle \trianglelefteq D$, a contradiction, therefore $H \cap Q_8 \neq \{1\}$ thus $|H \cap Q_8| = 2$ or 4. If $|H \cap Q_8| = 2$ is trivial. On the other hand, if $|H \cap Q_8| = 4$ then $H \cap Q_8 = H = \langle \pm\gamma \rangle$ for some $\gamma = i, j$ or k, where it follows that $a^2 = (\pm\gamma)^2 = -1$. $\square$

Corollary 3.6 *Let D be a Dedekindian 2-group containing Q_8 and consider E a non-trivial subgroup of D such that $E \cap Q_8 = \{1\}$. Then E has exponent 2 (E is an elementary abelian 2-group).*

Proof. Let $a \neq 1 \in E$ with $|a| = 2^k$, where $k \geq 2$. Clearly $b = a^{2^{k-2}} \in E$ and $|b| = 4$. As $E \cap Q_8 = \{1\}$ have $\langle b \rangle \cap Q_8 = \{1\}$ which contradicts Lemma 3.5, thus $|a| = 2$. $\square$

Proposition 3.7 *Let D be a Hamiltonian 2-group. Then $D = Q_8 \times E$, where E is trivial or an elementary abelian 2-group.*

Proof. By Lemma 3.4 we can consider that Q_8 is a subgroup of D and in addition there is a maximal subgroup E of D with the property $E \cap Q_8 = \{1\}$, thus $Q_8 \cdot E = Q_8 \times E \leqslant D$. To prove the proposition it is enough to prove that if a belongs to D then a also belongs to $Q_8 \times E$.

Let $a \in D$ whose order is equal to 2. If $a \in Q_8$ or $a \in E$ it is trivial. If $E = \{1\}$ follows from its maximality that $\langle a \rangle \cap Q_8 \neq \{1\}$ thus $a \in Q_8 \times E$. Now suppose that $E \neq \{1\}$ and $a \in D \backslash (Q_8 \cup E)$. Consider $\langle a \rangle \cdot E = \langle a \rangle \times E$. Clearly $\langle a \rangle \times E \neq D$ because that $Q_8 \leq D$, thus by maximality of E exist $1 \neq b \in E$ such that $q = ab \in Q_8$. Therefore $a = qb^{-1} \in Q_8 \times E$.

Consider now $a \in D \backslash (Q_8 \cup E)$ such that $|a| = 4$. Again if $a \in Q_8$, it is trivial. By Lemma 3.5 have $a^2 = -1 \in Q_8$, moreover as D is Dedekind have that $\langle a \rangle \trianglelefteq D$ so if $s \in \{i, j, k\}$ have $a^s = a$ or $a^s = a^3$. Note that if $a^i = a^3$ and $a^j = a^3$ then $a^k = a^9 = a$. Testing all possibilities we noticed that one of the following equations at least is true: $a^i = a$ or $a^j = a$ or $a^k = a$. Assume without loss of generality that $a^i = a$ then $(ia)^2 = i^2 a^2 = 1$ so $|ia| = 2$. Then as we saw in the first part of this proof we have $ia \in Q_8 \times E$ and so $a \in Q_8 \times E$ too.

Let's now show that there is no element in group D whose order is greater than or equal to 8. Let $a \in D$ with $|a| = 2^k$, where $k > 3$, clearly $b = a^{2^{k-3}}$ satisfies $|b| = 8$. It is enough to show that such b do not exist in D. Put $a \in D$ with $|a| = 8$ then $\langle a^2 \rangle = \left\{id, a^2, a^4, a^6\right\}$. By Lemma 3.5 we have $a^4 = -1 \in Q_8$ and in addition for the second part of this proof we have $a^2 = s.m \in Q_8 \times E$. Clearly $s \neq \pm 1$ because that $|a^2| = 4$ a contradiction, so $s \in \{\pm i, \pm j, \pm k\}$. Suppose without loss of generality that $s = i$. Thus $(a^2)^j = -im = -a^2 = a^6$. As $\langle a \rangle \trianglelefteq D$ and $(a^2)^j = a^6$ we have $a^j \neq a$ and $a^j \neq a^5$. Therefore the only possibilities for a^j are $a^j = a^3$ or $a^j = a^7$. Suppose first that $a^j = a^3$ then $(ja)^2 = jaja = j^2 a^4 = 1$, now if $a^j = a^7$ then $(ja)^4 = jajajaja = j^2 a^8 jaja = j^3 aja = j^4 a^8 = 1$, thus $|ja| = 2$ or 4. Again for the same reasons as above $ja \in Q_8 \times E$ and so $a \in Q_8 \times E$. Then we prove that $D = Q_8 \times E$, in particular if E is trivial then $D = Q_8$. □

Let us now use the results seen in this section to prove the following proposition. This is the reciprocal for the Baer-Dedekind Theorem.

Proposition 3.8 *Let G be a Hamiltonian group, then $G \cong Q_8 \times E \times A$, where A is an abelian torsion group whose elements have odd order and E is an elementary abelian 2-group.*

Proof. By Lemma 2.2 and Proposition 3.1 have that G is a nilpotent torsion group whose class is at most two, so $tor(G) = G$. By 5.2.7 in [15] have

$$G = Dr_{p \in \mathbb{P}} T_p = D \times Dr_{p \in \mathbb{P} \backslash \{2\}} T_p$$

where Dr denotes the direct product of groups, T_p is the unique maximum p-subgroup of G and $\mathbb{P}$ is the set of all prime positive integers such that $D = T_2$. Now as T_p is a Dedekind group for each odd prime number p, follows from Proposition 3.2 that each T_p (odd p) is abelian which implies that $A = Dr_{p\in\mathbb{P}\setminus\{2\}} T_p$ is an abelian torsion group whose elements have odd order. Clearly D is not abelian otherwise G would be an abelian group. Then D is a Dedekindian non-abelian 2-group. By Proposition 3.7 have $D \cong Q_8 \times E$, where E is an elementary abelian 2-group. Therefore $G \cong Q_8 \times E \times A$. $\square$

We come to the main theorem whose original proof can be found in (for infinite groups, e.g, [3, 11, 19]), (for finite groups, e.g, [5]) and (for generalizations, e.g, [1, 2, 12, 18, 7, 14, 4, 13]).

Proof of the Main Theorem: It follows from Lemma 2.3 and Proposition 3.8. $\square$

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