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ON INITIAL BOUNDARY VALUE PROBLEM FOR A SEMILINEAR REACTION DIFFUSION EQUATION AND NON-LINEAR SOURCE TERM WITH BLOW-UP

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Abstract

In this paper, we consider the following initial-boundary value problem

$$(P)\begin{cases} u_t(x,t) = \varepsilon Lu(x,t) + f(u(x,t)) \quad \text{in} \quad \Omega\times(0,T), \\ \frac{\partial u(x,t)}{\partial N} + b(x,t)g(u(x,t)) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ε is a positive parameter, L is an elliptic operator, $b(x,t)\geq 0$ in $\partial\Omega\times\mathbb{R}_+$, $g\in C^1(\mathbb{R})$, g(0)=0, f(s) is a positive, increasing, convex function for positive values of s and $\int^{\infty}\frac{ds}{f(s)}<\infty$. Under some assumptions, we show that the solution of the above problem blows up in a finite time and its

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blow-up time goes to the one of the solution of the following differential equation

$$\begin{cases} \alpha^{'}(t) = f(\alpha(t)), & t > 0, \\ \alpha(0) = M, \end{cases}$$

as ε tends to zero where $M = \sup_{x\in\overline{\Omega}} u_0(x)$. We also extend the above result to other classes of nonlinear parabolic equations with nonlinear boundary conditions. Finally, we give some numerical results to illustrate our analysis.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value parabolic problem associated to the following differential equation

$$u_t(x,t) = \varepsilon Lu(x,t) + f(u(x,t)) \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u(x,t)}{\partial N} = -b(x,t)g(u(x,t)) \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{3}$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. The term f(s) is a positive, increasing, convex function for positive values of s, $\int^{\infty} \frac{ds}{f(s)} < +\infty$, $g(s) \in C^1(\mathbb{R})$, g(0)=0, b(x,t) is nonnegative and continuous in $\Omega \times \mathbb{R}_+$,

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{N} a_{ij}(x)\cos(\nu, x_j)\frac{\partial u}{\partial x_j},$$

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where the coefficient $a_{ij}(x) \in C^1(\Omega)$ satisfy the following inequality

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq C|\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^N, \quad C > 0, \quad a_{ij} = a_{ji},$$

ν is the exterior normal unit vector on $\partial\Omega$, ε is a positive parameter.
The initial data u_0 is a nonnegative and continuous function in Ω. Here (0,T)

is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T}\|u(x,t)\|_\infty = +\infty,$$

where $\|u(x,t)\|_\infty = \max_{x\in\bar{\Omega}}|u(x,t)|$. In this last case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

Definition 1.1. *we say that the solution u of (1.1)-(1.3) blows up in a finite time, if there exist a finite time T such that $\|u(\cdot,t)\|_\infty < \infty$ pour $t\in[0,T)$ but*

$$\lim_{t\to T}\|u(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega}|u(x,t)|$. and the time T is called the blow-up time of the solution u, when T is infinite, we say that the solution u exists globally.

Solutions of semilinear reaction diffusion equations which blow up in a finite time have been the subject of investigation of many authors (see [1], [2], [7],[9], [12], [13], [16-18], [23] and the references cited therein). In particular the above problem has been studied by a lot of authors and by standard methods based on the maximum principle, local existence, uniqueness, blow-up and global existence have been treated (see [11], [26]]).

In this paper, we are interesting in the asymptotic behavior of the blow-up time when ε is small enough.

Our work was motivated by the paper of Friedman and Lacey in [7], where they have considered the following initial-boundary value problem

$$u_t = \varepsilon\Delta u + g(u) \quad \text{in} \quad \Omega\times(0,T), \tag{4}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{5}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{6}$$

where $g(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^{+\infty}\frac{ds}{g(s)} < +\infty$. The initial datum u_0 is a positive and continuous function in $\overline{\Omega}$.

Where they have considered the problem (1.1)-(1.3) in the case where the operator L is replaced by the Laplacian and the nonlinear boundary conditions is replaced by the Dirichlet boundary conditions.

Under some additional conditions on the initial datum, they have shown that proved that when ε is small enough the solution u of (1.1)-(1.3) blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equation

$$\alpha^{'}(t) = f(\alpha(t)), \quad \alpha(0) = M, \tag{7}$$

as ε tends to zero where $M = \sup_{x\in\overline{\Omega}} u_0(x)$.
Let us notice that the result of Friedman and Lacey holds when f(0)¿0, but they have noticed that if the solution increases with respect to the second variable, it is possible that their result holds for f(0)=0. The proof in [7] is based on the construction of upper and lower solutions and it is difficult to extend the method in [7] to our problem. In this paper, we obtain a similar result for the problem described in (1.1)-(1.3) using both a modification of Kaplan's method (see [10]) and a method based on the construction of upper solutions. Our paper is written in the following manner. In the next section, under some conditions, we show that the solution u of (1.1)-(1.3) blows up in a finite time and its blow-up time goes to the one of the solution of the differential equation defined in (1.7) as ε goes to zero. In the third section, we extend the result of Section 2 to other classes of parabolic problems. Finally, in the last section, we give some numerical results to illustrate our analysis.

2 The blow-up solutions in reaction diffusion equation

In this section, under some assumptions, we show that the solution u of (1.1)-(1.3) blows up in a finite time and its blow-up time tends to the one of the solution of the differential equation defined in (1.7) as ε tends to zero. Before starting, let us recall a well known result. Consider the following eigenvalue problem

$$-L\varphi = \lambda\varphi \quad \text{in} \quad \Omega, \tag{8}$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega, \tag{9}$$

$$\varphi > 0 \quad \text{in} \quad \Omega. \tag{10}$$

It is known that the above problem has a solution (φ, λ) such that $\lambda > 0$ and we can normalize φ so that $\int_\Omega \varphi dx = 1$.
Now, let us state our first result on the blow-up.

Theorem 2.1. *Suppose that $u_0(x) = 0$ and f(0)¿0. Let ε be such that $\varepsilon < \frac{1}{A}$ where $A = \lambda \int_0^\infty \frac{ds}{f(s)}$. Then the solution u of (1.1)-(1.3) blows up in a finite time and its blow-up time T obeys the following relation*

$$T = T_0(1 + \varepsilon A) + o(\varepsilon) \quad as \quad \varepsilon \to 0 \tag{11}$$

where $T_0 = \int_0^\infty \frac{ds}{f(s)}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined in (1.7).

Proof. Since (0,T) is the maximal time interval of existence of the solution u, our aim is to show that T is finite and satisfies the above relation. The fact that the initial data u_0 is nonnegative in Ω implies that the solution u is also nonnegative in $\Omega \times (0, T)$ owing to the maximum principle. Let D be a domain such that $\bar{D} \subset \Omega$ and let w(x,t) be the solution of the following initial-boundary value problem.

$$w_t(x,t) - \varepsilon Lw(x,t) = f(w(x,t)) \quad \text{in} \quad D \times (0, T^*), \tag{12}$$

$$w(x,t) = 0 \quad \text{on} \quad \partial D \times (0, T^*), \tag{13}$$

$$w(x,0) = 0 \quad \text{in} \quad D, \tag{14}$$

where $(0, T^*)$ is the maximal time interval of existence of w. Consider the following eigenvalue problem

$$-L\psi = \lambda_D \psi \quad \text{in} \quad D, \tag{15}$$

$$\psi = 0 \quad \text{on} \quad \partial D, \tag{16}$$

$$\psi > 0 \quad \text{in} \quad D. \tag{17}$$

It is well known that the above eigenvalue problem has a solution (ψ, λ_D) such that $0 < \lambda_D < \lambda$. Without loss of generality, we may suppose that $\int_D \psi dx = 1$. Introduce the function v(t) defined as follows

$$v(t) = \int_D \psi(x) w(x,t) dx.$$

Take the derivative of v in t and use (2.5) to obtain

$$v'(t) = \varepsilon \int_D \varphi(x) Lw(x,t) dx + \int_D f(w)\varphi(x) dx.$$

Applying Green's formula, we arrive at

$$v'(t) = \varepsilon \int_D wL\varphi dx + \int_D f(w)\varphi dx.$$

It follows from (2.8) and Jensen's inequality that

$$v'(t) \quad \geq \quad -\varepsilon\lambda_D v(t) + f(v(t)) \geq -\varepsilon\lambda v(t) + f(v(t)),$$

because $0 < \lambda_D < \lambda$. Obviously, we have

$$v'(t) \geq f(v(t))\left(1 - \frac{\varepsilon\lambda v(t)}{f(v(t))}\right).$$

It is not hard to see that

$$\int_0^\infty \frac{ds}{f(s)} \geq \sup_{t\geq 0}\int_0^t \frac{ds}{f(s)} \geq \sup_{t\geq 0}\frac{t}{f(t)}$$

because f(s) is an increasing function for s¿0. We deduce that

$$v'(t) \geq (1-\varepsilon A)f(v(t)), \quad \text{for} \quad t \in (0, T^*).$$

This estimate may be rewritten sa follows

$$\frac{dv}{f(v)} \geq (1-\varepsilon A)dt \quad \text{for} \quad t \in (0, T^*).$$

Integrate the above inequality over $(0, T^*)$ to obtain

$$T^* \leq \frac{1}{1-\varepsilon A}\int_0^\infty \frac{ds}{f(s)}$$

which implies that the solution w blows up at the time T^* because the quantity on the right hand side of the above inequality is finite. Since the solution u is nonnegative in $\Omega \times (0, T)$, it is easy to see that

$$u_t - \varepsilon Lu - f(u) \geq w_t - \varepsilon Lw - f(w) \quad \text{in} \quad D \times (0, T^0),$$

$$u \geq w \quad \text{on} \quad D \times (0, T^0),$$

$$u(x, 0) \geq w(x, 0) \quad \text{in} \quad D,$$

where $T^0 = \min\{T, T^*\}$. It follows from the maximum principle that $u \geq w$ in $D \times (0, T^0)$. If $T > T^*$ then we have

$$\lim_{t\to T^*}\|u(x,t)\|_\infty = \infty$$

which is a contradiction. Consequently

$$T \leq T^* \leq \frac{1}{(1-\varepsilon A)}\int_0^\infty \frac{ds}{f(s)}. \tag{18}$$

On the other hand, setting

$$z(x, t) = \alpha(t) \quad \text{in} \quad \bar{\Omega} \times (0, T_0),$$

it is not hard to see that

$$\begin{cases} z_t(x,t) - \varepsilon Lz + f(z(x,t)) = 0 \quad \text{in} \quad \Omega \times (0,T_0), \\ \frac{\partial z}{\partial N} + b(x,t)g(z) \geq 0 \quad \text{on} \quad \partial\Omega \times (0,T_0), \\ z(x,0) = 0 \quad \text{in} \quad \Omega. \end{cases}$$

The maximum principle implies that $0 \leq u(x,t) \leq z(x,t) = \alpha(t)$ in $\Omega \times (0,T_*)$, where $T_* = \min\{T,T_0\}$. It follows that $T \geq T_0$. Indeed suppose that $T < T_0$ which implies that $0 \leq u(x,T) \leq \alpha(T) < +\infty$ which is a contradiction because (0,T) is the maximal time interval of existence of the solution u. We deduce that

$$T \geq T_0 = \int_0^{\infty} \frac{ds}{f(s)}. \tag{19}$$

Apply Taylor's expansion to obtain $\frac{1}{1-\varepsilon A} = 1 + \varepsilon A + o(\varepsilon)$. Use (2.11),(2.12) and the above relation to complete the rest of the proof. □
Now let us consider the case where the initial data is not null. We have the following result.

Theorem 2.2. *Let f(0)=0. Suppose that* $\sup_{x\in\overline{\Omega}} u_0(x) = M > 0$ *and let* ε *be such that* $\varepsilon < \frac{1}{A}$ *where* $A = \frac{\lambda M}{2f(\frac{M}{2})}$. *Then the solution u of (1.1)-(1.3) blows up in a finite time and its blow-up time T satisfies the following relation*

$$T = T_0(1+\varepsilon A) + \frac{\varepsilon}{f(\frac{M}{2})} + o(\varepsilon) \quad as \quad \varepsilon \to 0$$

where $T_0 = \int_M^{\infty} \frac{ds}{f(s)}$ *is the blow-up time of the solution* $\alpha(t)$ *of the differential equation defined in (1.7).*

Proof From the maximum principle, there exists $a \in \Omega$ such that $M = u_0(a)$. Since u_0 is continuous in Ω, there exists $\delta > 0$ such that

$$u_0(x) \geq M - \varepsilon \quad \text{in} \quad B(a,\delta)$$

for ε small enough, where $B(a,\delta) = \{x \in \Omega : \|x-a\|_\infty < \delta\}$ and $\overline{B(a,\delta)} \subset \Omega$. Consider the eigenvalue problem below

$$-L\psi = \lambda_\varepsilon \psi \quad \text{in} \quad B(a,\delta),$$

$$\psi = 0 \quad \text{on} \quad \partial B(a,\delta),$$

$$\psi > 0 \quad \text{in} \quad B(a,\delta).$$

The above problem has a solution $(\psi, \lambda_\varepsilon)$ where $0 < \lambda_\varepsilon < \lambda$ and we can normalize ψ so that $\int_{B(a,\delta)} \psi dx = 1$.
Let w(x,t) be the solution of the following initial-boundary value problem

$$\begin{cases} w_t(x,t) - \varepsilon L w(x,t) - f(w(x,t)) = 0 \quad \text{in} \quad B(a,\delta) \times (0,T^*) \\ w(x,t) = 0 \quad \text{on} \quad \partial B(a,\delta) \times (0,T^*), \\ w(x,0) = u_0(x) \quad \text{in} \quad B(a,\delta), \end{cases}$$

where $(0, T^*)$ is the maximal time interval of existence of the solution w. Introduce the function v(t) defined as follows

$$v(t) = \int_\Omega w(x,t)\psi(x)dx.$$

As in the proof of Theorem 2.1, we get

$$v'(t) \geq -\varepsilon\lambda_\varepsilon v(t) + f(v(t)) \geq -\varepsilon\lambda v(t) + f(v(t)),$$

because $0 < \lambda_\varepsilon < \lambda$. We deduce that

$$v^{'}(t) \geq f(v(t))\left(1 - \frac{\varepsilon\lambda v(t)}{f(v(t))}\right). \tag{20}$$

When t=0, we see that $v^{'}(0) > 0$. Therefore, we have $v^{'}(t) > 0$ for $t \in (0, T^*)$. Indeed let t_0 be the first t¿0 such that $v'(t) > 0$ for $t \in (0, t_0)$ but $v'(t_0) = 0$ which implies that

$$v^{'}(t_0) \geq f(v(t_0))\left(1 - \frac{\varepsilon\lambda v(t_0)}{f(v(t_0))}\right).$$

Since f(s) is a convex function for the positive values of s and f(0)=0, then $\frac{f(s)}{s}$ is an increasing function for the positive values of s. The fact that $v(t_0) \geq v(0) \geq M - \varepsilon \geq \frac{M}{2}$ implies that

$$0 = v^{'}(t_0) \geq f(v(t_0))\left(1 - \frac{\varepsilon\lambda M}{2f(\frac{M}{2})}\right) > 0$$

which is a contradiction. We deduce that $v(t) \geq v(0)$ for $t \in (0, T^*)$. Since $v(0) \geq M - \varepsilon \geq \frac{M}{2}$, we arrive at

$$v^{'}(t) \geq f(v(t))\left(1 - \frac{\varepsilon\lambda M}{2f(\frac{M}{2})}\right),$$

which implies that

$$v^{'}(t) \geq (1 - \varepsilon A) f(v(t)) \quad \text{in} \quad (0, T^*). \tag{21}$$

We observe that

$$\frac{dv}{f(v)} \geq (1 - \varepsilon A)dt.$$

Integrate this inequality over $(0, T^*)$ to obtain

$$T^* \leq \frac{1}{(1-\varepsilon A)} \int_{v(0)}^{\infty} \frac{ds}{f(s)} \leq \frac{1}{(1-\varepsilon A)} \int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)}.$$

This implies that the solution w blows up in a finite time because the quantity on the right hand side of the second inequality is finite. On the other hand by the maximum principle, we have $u \geq 0$ in $\Omega \times (0, T)$, which implies that

$$u_t - \varepsilon Lu - f(u) \geq w_t - \varepsilon Lw - f(w) \quad \text{in} \quad B(a, \delta) \times (0, T_*),$$

$$u \geq w \quad \text{on} \quad \partial B(a, \delta) \times (0, T_*),$$

$$u(x, 0) \geq w(x, 0) \quad \text{in} \quad B(a, \delta),$$

where $T_* = \min\{T, T^*\}$. It follows from the maximum principle that

$$u(x, t) \geq w(x, t) \quad \text{in} \quad B(a, \delta) \times (0, T_*),$$

which implies that

$$T \leq T^* \leq \frac{1}{1-\varepsilon A} \int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)}. \tag{22}$$

Indeed, suppose that $T > T^*$. We have $\|u(x, T^*)\|_\infty > \|w(x, T^*)\|_\infty = +\infty$. But this is a contradiction because $(0, T)$ is the maximal time interval of existence of the solution u. We observe that

$$\int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)} = \int_{M}^{\infty} \frac{ds}{f(s)} + \int_{M-\varepsilon}^{M} \frac{ds}{f(s)} \leq \int_{M}^{\infty} \frac{ds}{f(s)} + \frac{\varepsilon}{f(M-\varepsilon)}.$$

because f(s) is an increasing function for the positive values of s. The fact that $f(M - \varepsilon) \geq f(\frac{M}{2})$ implies that

$$\int_{M-\varepsilon}^{\infty} \frac{ds}{f(s)} \leq \int_{M}^{\infty} \frac{ds}{f(s)} + \frac{\varepsilon}{f(\frac{M}{2})}. \tag{23}$$

Setting $z(x, t) = \alpha(t)$ in $\bar{\Omega} \times (0, T_0)$, it is not hard to see that

$$\begin{cases} z_t - \varepsilon Lz - f(z) = 0 \quad \text{in} \quad \Omega \times (0, T_0), \\ \frac{\partial z}{\partial N} + b(x, t)g(z) \geq 0 \quad \text{on} \quad \partial\Omega \times (0, T_0), \\ z(x, 0) \geq u_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

The maximum principle implies that $0 \leq u(x,t) \leq z(x,t) = \alpha(t)$ in $\Omega \times (0, T^0)$ where $T^0 = \min\{T_0, T\}$. We deduce that

$$T \geq T_0 \int_M^{\infty} \frac{ds}{f(s)}. \tag{24}$$

Indeed, suppose that $T_0 > T$, which implies that $\alpha(T) \geq \|u(x,T)\|_\infty = +\infty$. But this is a contradiction because $(0, T_0)$ is the maximal time interval of existence of the solution $\alpha(t)$. Apply Taylor's expansion to obtain $\frac{1}{1-\varepsilon A} = 1 + \varepsilon A + o(\varepsilon)$. Use (2.15)-(2.16) and the above relation to complete the rest of the proof. □

Remark 2.1. *Theorem 2.2 remains valid when f(0)¿0 if we take* $A = \lambda \int_0^{\infty} \frac{ds}{f(s)}$. *Indeed using (2.13) and the fact that* $\int_0^{+\infty} \frac{ds}{f(s)} \geq \sup_{s \geq 0} \frac{s}{f(s)}$ *we obtain the inequality in (2.14). Now, reasoning as in the proof of Theorem 2.1, we obtain the desired result.*

3 Estimates Of The Other blow-up Times

In this section, we extend the previous results considering the following initial-boundary value problem

$$(\varphi(u))_t = \varepsilon L u + f(u) \quad \text{in} \quad \Omega \times (0,T), \tag{25}$$

$$\frac{\partial u}{\partial N} + b(x,t) g(u) = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{26}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{27}$$

where $\varphi(s)$ is a nonnegative and increasing function for the positive values of s. In addition $\int^{\infty} \frac{\varphi'(s)}{f(s)} < \infty$. We have the following results using the methods described in the proofs of the above theorems.

Theorem 3.1. *Let* $\frac{f(0)}{\varphi'(0)} > 0$. *Suppose that* $\varepsilon < \frac{1}{B}$ *where* $B = \lambda \int_0^{\infty} \frac{\varphi'(s)}{f(s)} ds$. *Then the solution u of (3.1)-(3.3) blows up in a finite time and its blow-up time T satisfies the following relation.*

$$T = T_0(1 + \varepsilon B) + o(\varepsilon) \quad as \quad \varepsilon \to 0$$

where $T_0 = \int_0^{\infty} \frac{\varphi'(s)}{f(s)} ds$ *is the blow-up time of the solution* $\alpha(t)$ *of the differential equation defined as follows*

$$\begin{cases} \varphi'(\alpha(t))\alpha'(t) = f(\alpha(t)), & t > 0, \\ \alpha(0) = M, \end{cases}$$

where $M = \sup_{x \in \overline{\Omega}} u_0(x)$.

Theorem 3.2. *Assume that* $\lim_{s\to 0}\frac{f(s)}{\varphi'(s)} = 0$. *Suppose that* $\sup_{x\in\Omega} u_0(x) = M > 0$ *and let* ε *be such that* $\varepsilon < \frac{1}{A}$ *where* $A = \frac{\lambda\varphi'(\frac{M}{2})}{f(\frac{M}{2})}$. *Then the solution* u *of (3.1)-(3.3) blows up in a finite time and its blow-up time* T *satisfies*

$$T = T_0(1+\varepsilon A) + \frac{\varepsilon\varphi'(\frac{M}{2})}{f(\frac{M}{2})} \quad as \quad \varepsilon \to 0$$

where $T_0 = \int_M^{+\infty}\frac{ds}{f(s)}$ *is the blow-up time of the solution* $\alpha(t)$ *of the differential equation defined as follows*

$$\begin{cases} \varphi'(\alpha(t))\alpha'(t) = f(\alpha(t)), & t > 0 \\ \alpha(0) = M, \end{cases}$$

where $M = \sup_{x\in\overline{\Omega}} u_0(x)$.

4 Numerical experiments

In this section, we consider the radial symmetric solution of the boundary value problem :

$$u_t = \varepsilon\Delta u + e^u \quad \text{in} \quad B\times(0,T),$$

$$\frac{\partial u}{\partial \nu} + u = 0 \quad \text{on} \quad S\times(0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad B,$$

where $B = \{x\in\mathbb{R}^N/|x| < 1,\}$, $S = \{x\in\mathbb{R}^N/|x| = 1,\}$. The above problem may be rewritten in the following form

$$u_t = \varepsilon\left(u_{rr} + \frac{N-1}{r}u_r\right) + e^u, \quad r\in(0,1), \quad t\in(0,T), \tag{28}$$

$$u_r(0,t) = 0, \quad u_r(1,t) + u(1,t) = 0, \quad t\in(0,T), \tag{29}$$

$$u(r,0) = u_0(r), \quad r\in(0,1). \tag{30}$$

Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (4.1)-(4.3) by the solution $U_h^{(n)} = (U_0^{(n)},\cdots,U_I^{(n)})^T$ of the following explicit scheme.

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \varepsilon N\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + e^{U_0^{(n)}},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{n}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)+e^{U_i^{(n)}}, 1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_{I-1}^{(n)}-2U_I^{(n)}}{h^2}-\varepsilon\left(\frac{2}{h}+(N-1)\right)U_I^{(n)}+e^{U_I^{(n)}},$$

$$U_i^{(0)}=\varphi_i,\quad 0\le i\le I,$$

where $\Delta t_n=\min(\frac{h^2}{4N\varepsilon}, h^2e^{-\|U^n\|_\infty})$ with $\|U_h^{n)}\|_\infty=\sup_{0\le i\le I}|U_i^n|$. Let us notice that the condition $\Delta t_n\le\frac{h^2}{4h\varepsilon}$ ensures the stability of the explicit scheme. We also approximate the solution u of (4.1)-(4.3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2}+e^{U_0^{(n)}},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon\left(\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h}\right)$$

$$+e^{U_i^{(n)}},\quad 1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_{I-1}^{(n+1)}-2U_I^{(n+1)}}{h^2}-\varepsilon\left(\frac{2}{h}+(N-1)\right)U_I^{(n+1)}+e^{U_I^{(n)}},$$

$$U_i^{(0)}=\varphi_i,\quad 0\le i\le I,$$

where $\Delta t_n=h^2e^{-\|U_h^n\|_\infty}$.

The explicit scheme may be written as follows:

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_1^{(n)}-2U_0^{(n)}}{h^2}+e^{U_0^{(n)}},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{n}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)+e^{U_i^{(n)}},$$

$$1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_{I-1}^{(n)}-2U_I^{(n)}}{h^2}-\varepsilon\left(\frac{2}{h}+(N-1)\right)U_I^{(n)}+e^{U_I^{(n)}},$$

$$U_i^{(0)}=\varphi_i,\quad 0\le i\le I,$$

or

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{n}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)$$

$$+e^{U_i^{(n)}},\quad 1\leq i\leq I-1,$$

$$U_i^{(n+1)}-U_i^{(n)}=\Delta t_n\varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)$$

$$+\Delta t_n e^{U_i^{(n)}},\quad 1\leq i\leq I-1,$$

or

$$U_i^{(n+1)}=U_i^{(n)}+\Delta t_n\varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)$$

$$+\Delta t_n e^{U_i^{(n)}},\quad 1\leq i\leq I-1,$$

or

$$U_i^{(n+1)}=U_i^{(n)}+\frac{\varepsilon\Delta t_n}{h^2}U_{i+1}^{(n)}-\frac{2\varepsilon\Delta t_n}{h^2}U_i^{(n)}+\frac{\varepsilon\Delta t_n}{h^2}U_{i-1}^{(n)}+\frac{\varepsilon(N-1)\Delta t_n}{2ih^2}U_{i+1}^{(n)}-\frac{\varepsilon(N-1)\Delta t_n}{2ih^2}U_{i-1}^{(n)}$$

$$+\Delta t_n e^{U_i^{(n)}},\ 1\leq i\leq I-1,$$

or

$$U_i^{(n+1)}=\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})U_{i+1}^{(n)}+(1-2\frac{\varepsilon\Delta t_n}{h^2})U_i^{(n)}+\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})U_{i-1}^{(n)},$$

$$+\Delta t_n e^{U_i^{(n)}},\quad 1\leq i\leq I-1,$$

For $i=0$, $U_0^{(n+1)}=U_0^{(n)}+\Delta t_n\varepsilon N\frac{2U_1^{(n)}-2U_0^{(n)}}{h^2}+e^{U_0^{(n)}}$, or

For $i=0$, $U_0^{(n+1)}=\Delta t_n\varepsilon N(1-\frac{2}{h^2})U_0^{(n)}+\frac{2\Delta t_n\varepsilon N}{h^2}U_1^{(n)}+\Delta t_n e^{U_0^{(n)}}$,

For $i=1$, $U_1^{(n+1)}=\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})U_2^{(n)}+(1-2\frac{\varepsilon\Delta t_n}{h^2})U_1^{(n)}+\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})U_0^{(n)}+\Delta t_n+\Delta t_n e^{U_1^{(n)}}$

or
For $i=1$,
$$U_1^{(n+1)} = \left((1-2\frac{\varepsilon\Delta t_n}{h^2})\right)U_1^{(n)} + \left(\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})\right)U_2^{(n)} + \left(\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})\right)U_0^{(n)} + \Delta t_n e^{U_1^{(n)}},$$
For $i=2$,
$$U_2^{(n+1)} = \left(\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})\right)U_1^{(n)} + \left((1-2\frac{\varepsilon\Delta t_n}{h^2})\right)U_2^{(n)} + \left(\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})\right)U_3^{(n)} + +\Delta t_n e^{U_2^{(n)}},$$
For $i=3$,
$$U_3^{(n+1)} = \left(\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})\right)U_2^{(n)} + \left((1-2\frac{\varepsilon\Delta t_n}{h^2})\right)U_3^{(n)} + \left(\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})\right)U_4^{(n)} + \Delta t_n e^{U_3^{(n)}},$$
...
For $i=I-1$,
$$U_{I-1}^{(n+1)} = \left(\frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i})\right)U_{I-2}^{(n)} + \left((1-2\frac{\varepsilon\Delta t_n}{h^2})\right) + \left(\frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i})\right)U_I^{(n)} + +\Delta t_n e^{U_I^{(n)}},$$
For $i=I$,
$$U_I^{(n+1)} = \left((\frac{2\varepsilon N\Delta t_n}{h^2})\right)U_{I-1}^{(n)} + \left((1-\frac{2\varepsilon N\Delta t_n}{h^2})\right)U_I^{(n)} - \varepsilon\Delta t_n\left(\frac{2}{h}+(N-1)\right)U_I^{(n)} + e^{U_I^{(n)}},$$
It leads us to the linear system below
$$U_i^{(n+1)} = AU_i^{(n)} + \left(F^{(n)}\right)_i$$
where A is a $I\times I$ tridiagonal matrix defined as follows
$$A = \begin{pmatrix} (1-2\frac{\varepsilon\Delta t_n}{h^2}) & \frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i}) & 0 & \cdots & 0 \\ \frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i}) & (1-2\frac{\varepsilon\Delta t_n}{h^2}) & \frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i}) & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{\varepsilon\Delta t_n}{h^2}(1+\frac{(N-1)}{2i}) \\ 0 & 0 & \cdots & \frac{\varepsilon\Delta t_n}{h^2}(1-\frac{(N-1)}{2i}) & (1-2\frac{\varepsilon\Delta t_n}{h^2}) \end{pmatrix}$$
It implies that
$$A = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_0 & a_0 & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_0 \\ 0 & 0 & \cdots & c_0 & a_0 \end{pmatrix}$$
with
$$a_0 = 1 - 2\frac{\varepsilon\Delta t_n}{h^2},$$

$$b_0 = \frac{\varepsilon \Delta t_n}{h^2}(1 + \frac{(N-1)}{2i}), \quad i = 1, ..., I-2,$$

$$c_0 = \frac{\varepsilon \Delta t_n}{h^2}(1 - \frac{(N-1)}{2i}), \quad i = 1, ..., I-1,$$

$\left(F^{(n)}\right)_i = \Delta t_n e^{U_i^{(n)}}$

and A a three-diagonal matrix verifying the following properties:
$A_{i,i} = 1 - 2\frac{\varepsilon \Delta t_n}{h^2} > 0, \quad 0 \le i \le I$ and $A_{i-1,i} = \frac{\varepsilon \Delta t_n}{h^2}(1 - \frac{(N-1)}{2i})$
$A_{i,i+1} = \frac{\varepsilon \Delta t_n}{h^2}(1 + \frac{(N-1)}{2i}), \quad 2 \le i \le I-2$ so that$A_{i,i} \ge \sum_{i \ne j} A_{i,j}$
It follows that $U_h^{(n)}$ exists for $n \ge 0$. In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n \ge 0$.

We need the following definition.

Definition 4.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if* $\lim_{n\to\infty} \|U_h^{(n)}\| = \infty$ *and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical blow-up time of the discrete solution $U_h^{(n)}$.*

Let us notice that in the explicit scheme,the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, existence and nonnegativity are also guaranteed by standard methods (see for instance [3]).

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical results for $U_i^{(0)} = 0$

First case: N=2; $\varepsilon = 1/10$

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	1.004277	8007	18	-
32	1.002690	30616	132	-
64	1.002264	116826	1011	1.90
128	1.002143	444769	7469	1.82

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	1.004375	8007	22	-
32	1.002714	30616	132	-
64	1.002270	116826	1464	1.90
128	1.002147	444770	7857	1.85

Second case: N=2; $\varepsilon = 1/100$

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	1.001954	7986	18	-
32	1.000488	30521	85	-
64	1.000122	116405	841	2.00
128	1.000030	442906	6976	1.99

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	1.001950	7988	17	-
32	1.000447	30531	88	-
64	1.000122	116446	768	2.00
128	1.000042	444535	68506	2.18

Third case: $N = 3$, $\varepsilon = 1/10$

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	1.005842	4508	11	-
32	1.004096	16625	71	-
64	1.003593	60869	500	1.80
128	1.003447	220034	3482	1.78

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	1.005982	4509	12	-
32	1.004130	16626	67s	-
64	1.003602	60869	713	1.81
128	1.003450	220036	3785	1.80

Fourth case: $N = 3$, $\varepsilon = 1/100$

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	1.001954	4509	12	-
32	1.000488	16626	67	-
64	1.000122	60869	713	2.00
128	1.000030	220036	3785	1.99

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	1.001950	4476	12	-
32	1.000447	16481	67	-
64	1.000122	60245	658	2.00
128	1.000030	218267	3600	1.99

Numerical results for $U_i^{(0)} = 20sin(i\pi h)$

.

Here the term of the source $e^{U_i^{(n)}}$ is replaced by $(U_i^{(n)})^2$. In this last case, we take

$$\Delta t_n = \min\{\frac{h^2}{4N\varepsilon}, \frac{h^2}{\|U_h^{(n)}\|_\infty}\},$$

for the explicit scheme and $\Delta t_n = \frac{h^2}{\|U_h^{(n)}\|_\infty}$ for the implicit scheme

First case: N=2; $\varepsilon = 1/10$

Table 9: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	0.0525112	3763	8	-
32	0.052437	16649	56	-
64	0.052401	49040	550	1.26
128	0.052323	180378	3423	1.22

Table 10: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	0.052527	3763	11	-
32	0.052437	16649	56	-
64	0.052401	49040	550	1.32
128	0.052310	181289	3567	1.34

Second case: N=2; $\varepsilon = 1/100$

Table 11: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	0.050426	3721	8	-
32	0.050289	13447	56	-
64	0.050256	48112	400	2.05
128	0.050248	180092	3580	2.04

Table 12: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	CPU_t	s
16	0.050428	3721	11	-
32	0.050290	13447	55	-
64	0.050256	48112	560	2.02
128	0.050247	180093	3357	1.92

Remark 4.1. *If we consider the problem (1.1)-(1.3) in the case where the initial data is null and the reaction term $f(u) = e^u$, it is not hard to see that the blow-up time of the solution of the differential equation defined in (1.7) equals one.*

We observe from Tables 1-8 that when ε diminishes, the numerical blow-up time decays to one. This result has been proved in Theorem 2.1.
When the initial data $u_0(x) = 20\sin(\pi x)$ and the reaction term $f(u) = u^2$, we find that the blow-up time of the solution of the differential equation defined in (1.7) equals 0.05.
We discover from Tables 9-12 that when ε diminishes, the numerical blow-up time decays to 0.05 which is a result proved in Theorem 2.3

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 4, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper.

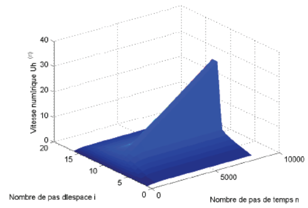

Figure 1: Evolution of the discrete solution, source $f(u) = e^u$, $\varepsilon = 1/10$, $\varphi_i = 0$, $I = 16$ (implicit scheme).

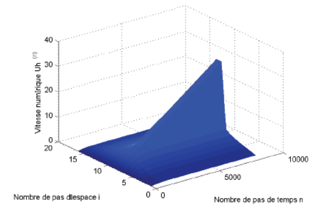

Figure 2: Evolution of the discrete solution, source $f(u) = e^u$, $\varepsilon = 1/10$, $\varphi_i = 0$, $I = 16$ (explicit scheme).

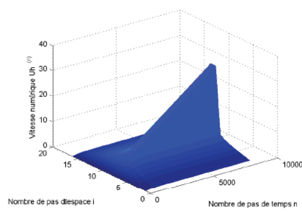

Figure 3: Evolution of the discrete solution, source $f(u) = e^u$, $\varepsilon = 1/100$, $\varphi_i = 0$, $I = 32$ (implicit scheme).

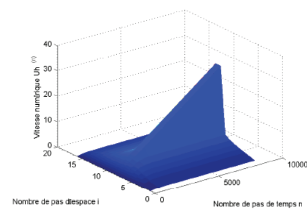

Figure 4: Evolution of the discrete solution, source $f(u) = e^u$, $\varepsilon = 1/100$, $\varphi_i = 0$,$I = 32$ (explicit scheme).

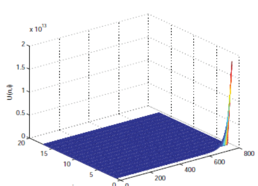

Figure 5: Evolution of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (implicit scheme).

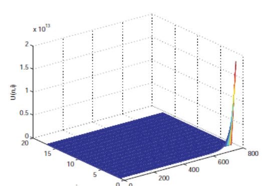

Figure 6: Evolution of the discrete solution, source $f(u) = u^2$, $N = 23$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (explicit scheme).

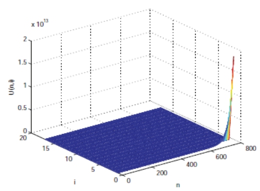

Figure 7: Evolution of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (implicit scheme).

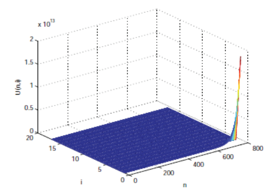

Figure 8: Evolution of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (explicit scheme).

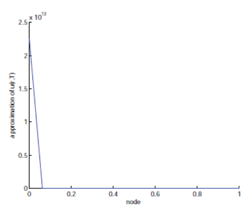

Figure 9: Profile of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (implicit scheme).

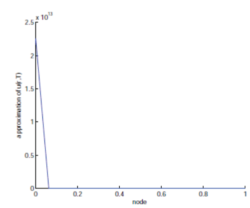

Figure 10: Profile of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 16$ (explicit scheme).

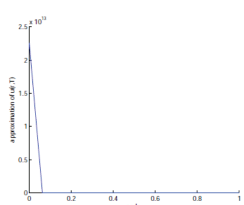

Figure 11: Profile of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 32$ (implicit scheme).

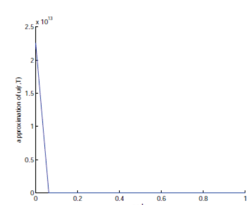

Figure 12: Profile of the discrete solution, source $f(u) = u^2$, $N = 2$; $\varepsilon = 1/10$, $\varphi_i = sin(i\pi h)$, $I = 32$ (explicit scheme).

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