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THE QUENCHING BEHAVIOR OF A NONLINEAR PARABOLIC EQUATION WITH RESPECT TO THE NON LINEAR SOURCE

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Abstract

The continuity of the quenching time is studied in this paper where we have considered a heat equation with variable reaction which quenches in a finite time. For this fact, we have estimated the quenching time and have proved that it is continuous as a function of the nonlinear source.

1 Introduction

Consider the following initial-boundary value problem

$$u_t = \Delta u - u^{-q} \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2}$$

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$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{3}$$

where $q > 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$. The initial datum $u_0 \in C^2(\overline{\Omega})$ and $u_0(x) > 0$ in $\overline{\Omega}$ and there exists a positive constant B such that

$$\Delta u_0(x) - (u_0(x))^{-p} \leq -B(u_0(x))^{-p} \quad \text{in} \quad \Omega. \tag{4}$$

Here $(0,T)$ is the maximal time interval of existence of the solution u, and by a solution, we mean the following.

Definition 1.1. *A solution of (1)–(3) is a function $u(x,t)$ continuous in $\overline{\Omega} \times [0,T)$, $u(x,t) > 0$ in $\overline{\Omega} \times [0,T)$, and twice continuously differentiable in x and once in t in $\Omega \times (0,T)$.*

The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a quenching in a finite time, namely

$$\lim_{t \to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x\in\overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u. Since the pioneering work of Kawarada in 1975 (see, [25]), the study of the phenomenon of quenching for semilinear heat equations has attracted a considerable attention (see, for example [2]-[4], [6]-[8], [11], [14], [22], [26], [28]-[30], [37-40] and the references cited therein). A typical example is the work in [7] where the problem (1)-(3) has been studied. Some authors have proved the existence and uniqueness of solution (see, [7], [16], [27]). This paper is the continuation of our work in [8] where we have considered the same problem. We have estimated the quenching time and studied its continuity as a function of the initial datum u_0. This time, the continuity of the quenching time as a function of the exponent of the nonlinear source is tackled. More precisely, we consider the following initial-boundary value problem

$$v_t = \Delta v - v^{-p(x)} \quad \text{in} \quad \Omega \times (0, T_h), \tag{5}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T_h), \tag{6}$$

$$v(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{7}$$

where $p \in C^0(\overline{\Omega})$, $\inf_{x\in\overline{\Omega}} p(x) = q > 0$, $p(x) = q + h(x)$ in Ω, $h(x) \geq 0$ in $\overline{\Omega}$. Here $(0, T_h)$ is the maximal time interval on which the solution v of (5)-(7)

exists. When T_h is finite, we say that the solution v of (5)-(7) quenches in a finite time and the time T_h is called the quenching time of the solution v. Consequently to the definition of the time T_h we have in this paper

$$v(x,t) > 0 \quad \text{in} \quad \overline{\Omega} \times (0, T_h).$$

If we set $g(x,u) = u^{-p(x)}$, then we observe that the function g is continuous in both variables and locally Lipschitz in the second one. Let us notice that, because the initial data of the different problems considered are sufficiently regular, the solutions of these problems exist and are regular. In addition, we note that the regularity of solutions is as important as the regularity of the initial data, and the maximum principle holds (see, [16], [27], [35]). In the present paper, we prove that if h is small enough, then the solution v of (5)-(7) quenches in a finite time and its quenching time T_h goes to T as h goes to zero where T is the quenching time of the solution u of (1)–(3). In addition we provide an upper bound of $|T_h - T|$ in terms of $\|h\|_\infty$. Similar results have been obtained in [5], [9], [17]-[21], [23], [24], [31], [32] where the authors have considered the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time).
This paper is structured as follows. In the following section, we show that under some assumptions, the solution v of (5)-(7) quenches in a finite time and estimate its quenching time. In the third section, we deal with the continuity of the quenching time and finally, in the last section, we give some numerical results to illustrate our analysis.

2 Quenching time

In this section, using an idea of Friedman and McLeod in [17], we may prove the following result on the quenching of the solution v of (5)-(7).

Theorem 2.1. *Suppose that there exists a constant $A \in (0,1]$ such that the initial datum at (7) satisfies*

$$\Delta u_0(x) - (u_0(x))^{-p(x)} \leq -A(u_0(x))^{-q} \quad \text{in} \quad \Omega. \tag{8}$$

Then, the solution v of (5)-(7) quenches in a finite time T_h which obeys the following estimate

$$T_h \leq \frac{(u_{0min})^{q+1}}{A(q+1)}.$$

Proof. We know that $(0, T_h)$ is the maximal time interval of existence of the solution v. Therefore, to prove our theorem, we have to show that T_h is finite and satisfies the above inequality. For this fact, we introduce $J(x,t)$ a function defined as follows

$$J(x,t) = v_t(x,t) + A(v(x,t))^{-q} \quad \text{in} \quad \overline{\Omega} \times [0, T_h).$$

A simple calculation yields

$$J_t - \Delta J = (v_t - \Delta v)_t - Aqv^{-q-1}v_t - A\Delta v^{-q} \quad \text{in} \quad \Omega \times (0, T_h). \tag{9}$$

It is not hard to see that $\Delta v^{-q} = q(q+1)v^{-q-2}|\nabla v|^2 - qv^{-q-1}\Delta v$ in $\Omega \times (0, T_h)$, which implies that $\Delta v^{-q} \geq -qv^{-q-1}\Delta v$ in $\Omega \times (0, T_h)$. Applying this inequality in (9), we find that

$$J_t - \Delta J \leq (v_t - \Delta v)_t - Aqv^{-q-1}(v_t - \Delta v) \quad \text{in} \quad \Omega \times (0, T_h). \tag{10}$$

Use (5) and (10) to obtain

$$J_t - \Delta J \leq p(x)v^{-p(x)-1}v_t + Aqv^{-q-p(x)-1} \quad \text{in} \quad \Omega \times (0, T_h).$$

Due to the fact that $q \leq p(x)$ in Ω, we discover that

$$J_t - \Delta J \leq p(x)v^{-p(x)-1}(v_t + Av^{-q}) \quad \text{in} \quad \Omega \times (0, T_h).$$

Making use of the expression of J, we derive the following inequality

$$J_t - \Delta J \leq p(x)v^{-p(x)-1}J \quad \text{in} \quad \Omega \times (0, T_h).$$

The boundary condition (5) allow us to write

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial v}{\partial \nu}\right)_t - Aqv^{-q-1}\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T_h).$$

According to (8), we have

$$J(x,0) = \Delta u_0(x) - (u_0(x))^{-p(x)} + A(u_0(x))^{-q} \leq 0 \quad \text{in} \quad \Omega.$$

One concludes by the maximum principle that $J(x,t) \leq 0$ in $\Omega \times (0, T_h)$, that is

$$v_t(x,t) + A(v(x,t))^{-q} \leq 0 \quad \text{in} \quad \Omega \times (0, T_h). \tag{11}$$

This estimate may be rewritten as follows

$$v^q dv \leq -Adt \quad \text{in} \quad \Omega \times (0, T_h). \tag{12}$$

Integrate the above inequality over $(0, T_h)$ to obtain

$$T_h \leq \frac{(v(x,0))^{q+1} - (v(x,T_h))^{q+1}}{A(q+1)} \quad \text{for} \quad x \in \Omega.$$

Employing (11), we observe that v is nonincreasing with respect to the second variable, which implies that $0 < v(x, T_h) \le v(x, 0)$ in Ω. We deduce that

$$T_h \le \frac{(v(x,0))^{q+1}}{A(q+1)} \quad \text{for} \quad x \in \Omega,$$

which implies that

$$T_h \le \frac{(u_{0min})^{q+1}}{A(q+1)}.$$

We observe that the quantity on the right hand side of the above inequality is finite. Consequently, v quenches at the time T_h and the proof is finished. □

Remark 2.1. *Let $t_0 \in (0, T_h)$. Integrating the inequality (12) from t_0 to T_h, we get*

$$T_h - t_0 \le \frac{(v(x,t_0))^{q+1}}{A(q+1)} \quad \textit{for} \quad x \in \Omega.$$

We deduce that

$$T_h - t_0 \le \frac{(v_{min}(t_0))^{q+1}}{A(q+1)}.$$

Remark 2.2. *In view of the condition (4) and reasoning as in the proof of Theorem 2.1, it is not hard to see that there exists a positive constant C such that $u_{\min}(t) \ge C(T-t)^{\frac{1}{q+1}}$ for $t \in (0, T)$.*

Before dealing with the continuity, we also need to show an upper bound of $u_{\min}(t)$ for $t \in (0, T)$. For this end, we state the theorem below.

Theorem 2.2. *Let u be the solution of (1)–(3). Then, there exists a positive constant B such that the following estimate holds*

$$u_{\min}(t) \le D(T-t)^{\frac{1}{1+p_+}} \quad \textit{for} \quad t \in (0, T), \tag{13}$$

where $p_+ = \max_{x\in\overline{\Omega}} p(x)$.

Proof. Since we want to provide an upper bound of $u_{\min}(t)$ for $t \in (0, T)$, we begin our proof by setting

$$w(t) = \frac{u_{\min}(t)}{\|u_0\|_\infty} \quad \text{for} \quad t \in [0, T).$$

Let t_1, $t_2 \in [0, T)$. Then there exist x_1, $x_2 \in \Omega$ such that $w(t_1) = \frac{u(x_1,t_1)}{\|u_0\|_\infty}$ and $w(t_2) = \frac{u(x_2,t_2)}{\|u_0\|_\infty}$. Use Taylor's expansion to establish

$$w(t_2) - w(t_1) \ge \frac{u(x_2,t_2) - u(x_2,t_1)}{\|u_0\|_\infty} = (t_2 - t_1)\frac{u_t(x_2,t_2)}{\|u_0\|_\infty} + o(t_2 - t_1),$$

$$w(t_2)-w(t_1)\le\frac{u(x_1,t_2)-u(x_1,t_1)}{\|u_0\|_\infty}=(t_2-t_1)\frac{u_t(x_1,t_1)}{\|u_0\|_\infty}+o(t_2-t_1),$$

which implies that $w(t)$ is Lipschitz continuous. Moreover, if $t_2>t_1$, then

$$\begin{aligned}\frac{w(t_2)-w(t_1)}{t_2-t_1} &\ge \frac{u_t(x_2,t_2)}{\|u_0\|_\infty}+o(1)\\ &= \frac{\Delta u(x_2,t_2)}{\|u_0\|_\infty}-\|u_0\|_\infty^{-p(x_2)-1}\left(\frac{u(x_2,t_2)}{\|u_0\|_\infty}\right)^{-p(x_2)}+o(1).\end{aligned}$$

Exploiting the maximum principle, we know that $u(x,t)\le\|u_0\|_\infty$ in $\Omega\times(0,T)$. This implies that $-\left(\frac{u(x_2,t_2)}{\|u_0\|_\infty}\right)^{-p(x_2)}\ge-\left(\frac{u(x_2,t_2)}{\|u_0\|_\infty}\right)^{-p_+}$. It follows that

$$\frac{w(t_2)-w(t_1)}{t_2-t_1}\ge\frac{\Delta u(x_2,t_2)}{\|u_0\|_\infty}-\beta\left(\frac{u(x_2,t_2)}{\|u_0\|_\infty}\right)^{-p_+}+o(1),$$

where $\beta=\max\{\|u_0\|_\infty^{-q-1},\|u_0\|_\infty^{-p_+-1}\}$. Letting $t_2\to t_1$, and using the fact that $\Delta u(x_2,t_2)\ge 0$, we obtain $w'(t)\ge-\beta(w(t))^{-p_+}$ for a.e. $t\in(0,T)$. This inequality can be rewritten as follows $w^{p_+}dw\ge-\beta dt$ for a.e. $t\in(0,T)$. Integrate the above inequality over (t,T) to obtain $\beta(T-t)\ge\frac{(w(t))^{1+p_+}}{1+p_+}$ for $t\in(0,T)$. Since $w(t)=\frac{u_{\min}(t)}{\|u_0\|_\infty}$, we arrive at

$$u_{\min}(t)\le\|u_0\|_\infty\left(\beta(1+p_+)(T-t)\right)^{\frac{1}{1+p_+}}\quad\text{for}\quad t\in(0,T).$$

This estimate ends the proof when we set $\|u_0\|_\infty\left(\beta(1+p_+)\right)^{\frac{1}{1+p_+}}=D$. □

3 Continuity of the quenching time

In this section, we shall present our main result which consists in proving an upper bound of $|T_h-T|$ in terms of $\|h\|_\infty$ by the following theorem.

Theorem 3.1. *Suppose that the problem (1)–(3) has a solution u which quenches at the time T. Then, under the assumption of Theorem 2.1, the solution v of (5)–(7) quenches in a finite time T_h, and there exist positive constants α, b, μ and γ such that for h small enough, the following estimate holds*

$$|T_h-T|\le\alpha\left(\ln(\mu+\frac{b}{\|h\|_\infty})\right)^{-\gamma}.$$

Proof. According to Theorem 2.1, the solution v quenches in a finite time T_h. In order to prove the above estimate, we proceed as follows. Let $T^*=\min\{T,T_h\}$ and introduce the error function $e(x,t)$ defined as follows

$$e(x,t)=v(x,t)-u(x,t)\quad\text{in}\quad\overline{\Omega}\times[0,T^*).$$

Let $t_0 \in (0, T^*)$. It is easy to establish by the mean value theorem that

$$e_t - \Delta e = p(x)\theta^{-p(x)-1}e - \ln(v)v^{-s(x)}h \quad \text{in} \quad \Omega \times (0, t_0), \tag{14}$$

$$\frac{\partial e}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, t_0), \tag{15}$$

$$e(x, 0) = 0 \quad \text{in} \quad \overline{\Omega}, \tag{16}$$

where θ lies between u and v, and $s(x)$ between q and $p(x)$. Using the fact that $\ln(\sigma) \leq \sigma$ for $\sigma > 0$, the equality (14) can be rewritten as follows

$$e_t - \Delta e \leq p(x)\theta^{-p(x)-1}e + v^{-s(x)-1}h \quad \text{in} \quad \Omega \times (0, t_0).$$

A transformation gives

$$\begin{aligned} e_t - \Delta e \quad &\leq \quad p(x)\|u_0\|_\infty^{-p(x)-1}\left(\frac{\theta}{\|u_0\|_\infty}\right)^{-p(x)-1} e \\ &+ \quad \|u_0\|_\infty^{-s(x)-1}\left(\frac{v}{\|u_0\|_\infty}\right)^{-s(x)-1} h \quad \text{in} \quad \Omega \times (0, t_0). \end{aligned}$$

According to the maximum principle, it is easy to see that $\frac{v}{\|u_0\|_\infty} \leq 1$ and $\frac{\theta}{\|u_0\|_\infty} \leq 1$ in $\Omega \times (0, t_0)$. Due to the fact that the function $x \to A^{-x}$ ($A \in (0, 1)$) is nondecreasing for $x \in (0, \infty)$, the following estimate holds

$$e_t - \Delta e \leq p_+C_0\left(\frac{\theta}{\|u_0\|_\infty}\right)^{-p_+-1}|e| + C_0\left(\frac{v}{\|u_0\|_\infty}\right)^{-p_+-1} h \quad \text{in} \quad \Omega \times (0, t_0), \tag{17}$$

where $C_0 = \max\{\|u_0\|_\infty^{-q-1}, \|u_0\|_\infty^{-p_+-1}\}$. Using Remarks 2.1 and 2.2, there exist positive constants C and C_1 such that for $t \in (0, t_0)$,

$$u_{\min}(t) \geq C(T - t)^{\frac{1}{q+1}} \quad \text{and} \quad v_{\min}(t) \geq C_1(T_h - t)^{\frac{1}{q+1}}.$$

There exists a positive constant C_2 such that $\min\{C(T - t)^{\frac{1}{q+1}},\ C_1(T_h - t)^{\frac{1}{q+1}}\} = C_2(T - t)^{\frac{1}{q+1}}$. Then, we have $\theta(x, t) \geq C_2(T - t)^{\frac{1}{q+1}}$ in $\Omega \times (0, t_0)$. Applying these estimates in (17), we have

$$e_t \leq \Delta e + \frac{C_3}{(T - t)^{\frac{1+p_+}{q+1}}}|e| + \frac{C_4 h}{(T - t)^{\frac{1+p_+}{q+1}}} \quad \text{in} \quad \Omega \times (0, t_0),$$

where $C_3 = p_+C_0\left(\frac{C_2}{\|u_0\|_\infty}\right)^{-p_+-1}$ and $C_4 = C_0\left(\frac{C_2}{\|u_0\|_\infty}\right)^{-p_+-1}$. Consider the following ODE

$$Z'(t) = \frac{C_3 Z(t)}{(T - t)^\delta} + \frac{C_4 h}{(T - t)^\delta} \quad \text{for} \quad t \in (0, t_0), \quad Z(0) = 0,$$

where $\delta = \frac{1+p_+}{q+1}$. Its solution $Z(t)$ is given explicitly by

$$Z(t) = \frac{C_4}{C_3} C_5 h e^{\frac{C_3}{\delta-1}(T-t)^{1-\delta}} - \frac{C_4}{C_3} h \quad \text{for} \quad t \in [0, t_0),$$

where $C_5 = e^{\frac{-C_3}{\delta-1}T^{1-\delta}}$. An application of the maximum principle gives

$$e(x,t) \leq Z(t) = \frac{C_4}{C_3} h \left(C_5 e^{\frac{C_3}{\delta-1}(T-t)^{1-\delta}} - 1 \right) \quad \text{in} \quad \Omega \times [0, t_0).$$

Fix a a positive constant and let $t_1 \in (0, T^*)$ be a time such that $\|e(\cdot, t_1)\|_\infty \leq \frac{C_4}{C_3} \|h\|_\infty \left(C_5 e^{\frac{C_3}{\delta-1}(T-t_1)^{1-\delta}} - 1 \right) = a$ for h small enough. This implies that

$$T - t_1 = \left(\frac{\delta - 1}{C_3} \ln\left(\frac{1}{C_5} + \frac{C_3 a}{C_4 C_5 \|h\|_\infty}\right) \right)^{\frac{1}{1-\delta}}. \tag{18}$$

On the other hand, by Remark 2.1 and the triangle inequality, we have

$$|T_h - t_1| \leq \frac{(v_{\min}(t_1))^{q+1}}{A(q+1)} \leq \frac{(u_{\min}(t_1) + \|e(\cdot, t_1)\|_\infty)^{q+1}}{A(q+1)}.$$

Using Theorem 2.2 and the fact that $\|e(\cdot, t_1)\|_\infty \leq a$, we obtain

$$|T_h - t_1| \leq \frac{\left(D(T - t_1)^{\frac{1}{1+p_+}} + a \right)^{q+1}}{A(q+1)}. \tag{19}$$

We can find a positive constant C_6 such that

$$D(T - t_1)^{\frac{1}{1+p_+}} + a = C_6 (T - t_1)^{\frac{1}{1+p_+}}.$$

Applying the above equality in (18) we obtain that

$$|T_h - t_1| \leq C_7 |T - t_1|^{\frac{q+1}{1+p_+}},$$

where $C_7 = \frac{C_6^{q+1}}{A(q+1)}$. We deduce from the above estimate and the triangle inequality that

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq |T - t_1| + C_7 |T - t_1|^{\frac{q+1}{1+p_+}}.$$

This implies that there exists a positive constant C_8 such that

$$|T - T_h| \leq C_8 |T - t_1|^{\frac{q+1}{1+p_+}}.$$

Since h is small enough, we have $\ln(\frac{1}{C_5} + \frac{C_3 a}{C_4 C_5 \|h\|_\infty}) \geq 0$. Using the equality (18) and the fact that $1 - \delta \leq 0$, we see that, there exist positive constants α, b, μ and γ such that

$$|T - T_h| \leq \alpha \left(\ln\left(\mu + \frac{b}{\|h\|_\infty}\right) \right)^{-\gamma}.$$

This ends the proof. □

4 Numerical results

To compute the numerical results we need to consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \Delta u - u^{-p(x)} \quad \text{in} \quad B \times (0, T),$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad S \times (0, T),$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad B,$$

where $p(x) = \psi(|x|)$, $u_0(x) = \varphi(|x|)$, $B = \{x \in \mathbb{R}^N;\ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N; \|x\| = 1\}$. Another form of the above problem is

$$u_t = u_{rr} + \frac{N-1}{r} u_r - u^{-\psi(r)}, \quad r \in (0, 1), \quad t \in (0, T), \tag{20}$$

$$u_r(0, t) = 0, \quad u_r(1, t) = 0, \quad t \in (0, T), \tag{21}$$

$$u(r, 0) = \varphi(r), \quad r \in [0, 1], \tag{22}$$

where, we take $\psi(r) = 1 + \frac{\varepsilon r}{r+1}$ with $\varepsilon \in [0, 1]$ and $\varphi(r) = 4 + 3\cos(\pi r)$. In order to compute the numerical solution, we need to construct an adaptive scheme. For this fact, define the grid $x_i = ih$, $0 \leq i \leq I$ where I is a positive integer and $h = 1/I$. Approximate the solution u of (20)-(22) by the solution $U_h^{(n)} = (U_0^{(n)}, ..., U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - (U_0^{(n)})^{-\psi_0},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}$$

$$-(U_i^{(n)})^{-\psi_i}, \quad 1 \leq i \leq I - 1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = N \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - (U_I^{(n)})^{-\psi_I},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $\psi_i = 1 + \frac{\varepsilon ih}{ih+1}$ and $\varphi_i = 4 + 3\cos(\pi ih)$. For the time step we take

$$\Delta t_n = \min\{\frac{(1-h^2)h^2}{2N}, h^2(U_{hmin}^{(n)})^{p_+ +1}\}$$

with $U_{hmin}^{(n)} = \min_{0\le i\le I} U_i^{(n)}$. This condition permits to the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, and ensures the positivity of the discrete solution. An important fact concerning the phenomenon of quenching is that, if the solution u quenches at the time T, then, when the time t approaches the quenching time T, the solution u decreases to zero rapidly. We also approximate the solution u of (20)-(22) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{-\psi_0 - 1}U_0^{(n+1)}$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}$$

$$-(U_i^{(n)})^{-\psi_i - 1}U_i^{(n+1)}, \quad 1 \le i \le I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = N\frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} - (U_I^{(n)})^{-\psi_I - 1}U_I^{(n+1)},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I.$$

As in the case of the explicit scheme, here again, we have transformed our scheme to an adaptive one by choosing $\Delta t_n = h^2(U_{hmin}^{(n)})^{1+p_+}$.
Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution is also guaranteed using standard methods (see for instance [6]). It is not hard to see that $u_{xx}(1,t) = \lim_{r\to 1}\frac{u_r(r,t)}{r}$ and $u_{xx}(0,t) = \lim_{r\to 0}\frac{u_r(r,t)}{r}$. Hence, if $r=0$ and $r=1$, we see that

$$u_t(0,t) = Nu_{rr}(0,t) - u^{-p}(0,t), \quad t \in (0,T),$$

$$u_t(1,t) = Nu_{rr}(1,t) - u^{-p}(1,t), \quad t \in (0,T).$$

These observations have been taken into account in the construction of our schemes when $i=0$ and $i=I$. We need the following definition.

Definition 4.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to\infty} U_{hmin}^{(n)} = 0$ *and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the numerical quenching time of the discrete solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $\psi_i = 1 + \frac{\varepsilon ih}{1+ih}$, $N = 2$
First case: $\varepsilon = 0$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	3.604286	5415	12	-
32	3.731558	21476	71	-
64	3.796654	84141	523	0.97
128	3.828011	335561	3782	1.04

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	3.604107	5325	13	-
32	3.731511	21121	87	-
64	3.796641	84721	1106	0.97
128	3.830302	331834	7718	0.95

Second case: $\varepsilon = 1/50$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	3.617938	5921	3	-
32	3.746080	23518	15	-
64	3.811621	92338	152	0.97
128	3.844694	360217	3684	0.99

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	3.617757	5921	5	-
32	3.746033	23518	26	-
64	3.811608	92338	310	0.97
128	3.844691	360217	8513	0.99

Third case: $\varepsilon = 1/1000$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	3.604966	5914	2	-
32	3.732282	23480	15	-
64	3.797401	92161	152	0.97
128	3.830262	359454	6284	0.99

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t_n	n	CPU time	s
16	3.604787	5914	4	-
32	3.732235	23480	25	-
64	3.797389	92161	307	0.97
128	3.830260	359454	7247	0.99

Remark 4.1. *If we consider the problem (20)-(22) in the case where the exponent of the nonlinear source* $\psi(r) = 1 + \frac{\varepsilon r}{1+r}$ *with* $\varepsilon = 0$*, and the initial datum* $\varphi(r) = 4 + 3\cos(\pi r)$*, we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is slightly equal to that in which the exponent of the nonlinear source increases slightly, that is when* ε *is a small positive real (see, Tables 1-6 for an illustration). This result confirms the theory established in the previous section.*

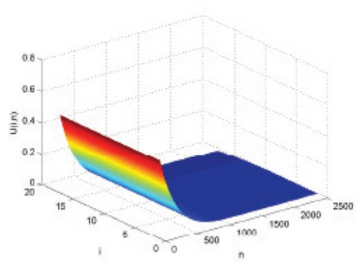

Figure 1: Evolution of the discrete solution, $f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 16$, $\varepsilon = 1/1000$ (explicit scheme).

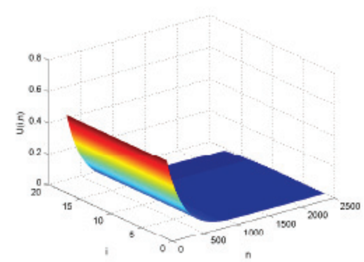

Figure 2: Evolution of the discrete solution, $f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 16$, $\varepsilon = 1/1000$ (implicit scheme).

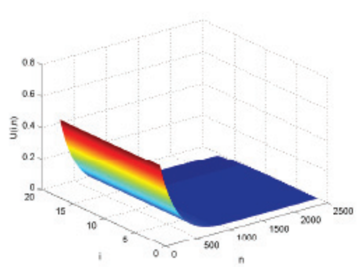

Figure 3: Evolution of the discrete solution, $f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 32$, $\varepsilon = 1/1000$ (explicit scheme).

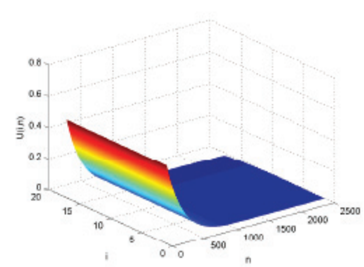

Figure 4: Evolution of the discrete solution, $f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 32$, $\varepsilon = 1/1000$ (implicit scheme).

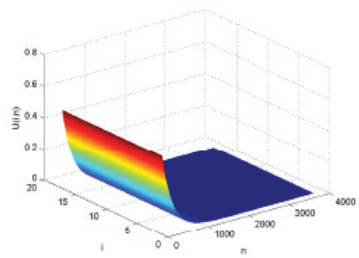

Figure 5: Evolution of the discrete solution,$f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 16$, $\varepsilon = 1/50$ (explicit scheme).

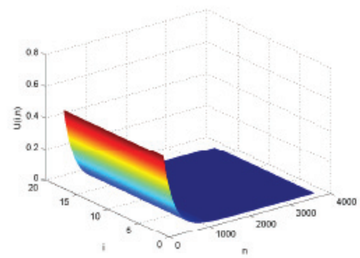

Figure 6: Evolution of the discrete solution,$f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 16$, $\varepsilon = 1/50$ (implicit scheme).

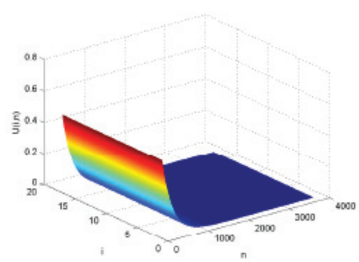

Figure 7: Evolution of the discrete solution,$f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 32$, $\varepsilon = 1/50$ (explicit scheme).

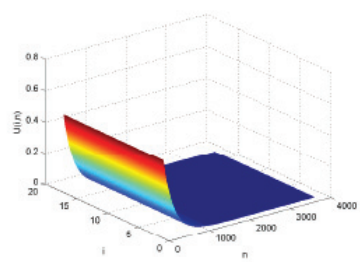

Figure 8: Evolution of the discrete solution,$f(U_k^{(n)}) = (U_k^{(n)})^{-p}$, $I = 32$, $\varepsilon = 1/50$ (implicit scheme).

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