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CLOSED DIVISOR GRAPHS AND ITS LABELINGS

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Abstract

In this paper, we defined a new graph labelling and named it as *closed divisor graph*. Let $G = (V, E)$ be a finite undirected (nonempty) graph. A graph G is said to be a *closed divisor graph* if there exists a vertex labeling using a set S of positive integers, called a *quotients of G*, such that two vertices are adjacent if and only if one vertex divides the other and their quotient is in the set S. In this paper, we study the properties of closed divisor graph and construct closed divisor labeling for several basic simple graphs and special graphs. We also proved that the closed divisor graph is a generalization of proper monograph.

1 Introduction

A graph labeling is an assignment of integers to the vertices, edges, or both the vertices and edges, such that some conditions must be satisfied [2]. Over the years, several graph labelings were discovered, studied and improved such as *proper monograph* [5, 1] and *divisor graph* [6, 3, 4]. Some of graph labelings came from existing graph labelings and some of them were newly discovered. The study of graph labelings plays a vital role in the field of Graph theory for its used in many applications like coding theory, crystallography, and communication networks.

Key words: Closed Divisor Graph, Divisor Graph, Graph Labeling, Proper Monograph.
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In this paper, all graphs are considered finite, simple and undirected. A graph G may be written as $G(V, E)$, where V and E are sets of vertices and edges of G. Let S be a finite non-empty set of positive integers. The *divisor graph* $G(S)$ of a finite subset $S \subset \mathbb{Z}$ is the graph $G(V, E)$ where $V = S$ and $uv \in E$ if and only if either $u|v$ or $v|u$[3]. In this study, our goal is to define a new graph labeling by adding one condition in the definition of the divisor graph. This labeling will be called as the *closed divisor graph*. The *closed divisor graph* $G(S)$ of a finite subset $S \subset \mathbb{Z}$ is the graph $G(V, E)$ where $V = S$ and $uv \in E$ if and only if either $u|v$ and $\frac{v}{u} \in S$, or $v|u$ and $\frac{u}{v} \in S$.

It is important to note that if G is a closed divisor graph, then it is not necessarily a divisor graph. For example, the cycle C_{2n+1} for $n \geq 2$ is not a divisor graph as shown by [3]. However, in this study, we have proven that C_n, for $n \geq 3$ is a closed divisor graph.

In this paper, we give a quotient construction for the simple graph such as the path, cycle, and complete graph. We also consider some graphs such as friendship graph, windmill, $(m, n)-$Tadpole, and $(m, n)-$lollipop. We have proven that if G is a proper monograph, then G is a closed divisor graph while the converse is not always true.

2 Special Families of Closed Divisor Graphs

This section contains the families of graphs that are closed divisor graphs. All graphs used are simple and connected.

Definition 2.1. A path, denoted as P_n, with n vertices, is a graph with exactly 2 vertices of degree 1 and $n - 2$ vertex/vertices of degree 2.

Theorem 2.1. *Every path P_n, $n \geq 2$ is a closed divisor graph.*

Proof. Let P_n be a path with $n \geq 2$ vertices.

$V(P_n) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, n-1$

$S := \{s_i\}_{i=0}^{n-1}$.

Define a labeling $\phi : V(P_n) \to S$, where

$$\phi(v_i) = s_i = q^{2^i}, i = 0, 1, 2, \cdots, n-1, \text{ for any fixed } q \geq 2,\ q \in \mathbb{Z}.$$

By the labeling ϕ, each s_i and s_{i+1} are connected since there exists a positive integer $e \in S$ such that $e \cdot q^{2^i} = q^{2^{i+1}}, e = q^{2^i}$. But for any $k > i+1$, s_i and s_k are not connected since $e \cdot q^2 = q^{2^k}, e = q^{(2^i)(2^{k-1}-1)} = q^{(2^{k-1}-1)} \notin S$. Hence, $d(s_0) = d(s_{n-1}) = 1, d(s_1) = d(s_2) = \cdots = d(s_{n-2}) = 2$. □

Definition 2.2. A cycle, denoted as C_n, with n vertices, is a graph where each vertex is of degree 2.

Figure 1: A closed divisor labeling of a path graph P_3.

Theorem 2.2. *Every cycle C_n, $n \geq 3$ is a closed divisor graph.*

Proof. Let C_n be a cycle with $n \geq 3$ vertices.
$V(C_n) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, n-1;\quad S := \{s_i\}_{i=0}^{n-1}$.

Define a labeling $\phi : V(C_n) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} q^{2^i} & \text{if } i = 0, 1, 2, \cdots, n-2 \text{ for any fixed } q \geq 2, \text{ q is a positive integer;} \\ q \cdot q^{2^{n-2}} & \text{if } i = n-1; \end{cases}$$

With the vertex labels defined above, by Theorem 2.1, s_i and s_{i+1} are connected for all $i = 0, 1, \cdots, n-3$. Moreover, $d(s_0), d(s_{n-2}) \geq 1$ and $d(s_i) \geq 2$, for all $i = 1, 2, \cdots, n-3$. Now, since $\dfrac{s_{n-1}}{s_0} = \dfrac{q \cdot q^{2^{n-2}}}{q} = q^{2^{n-2}} \in S$ and $\dfrac{s_{n-1}}{s_{n-2}} = \dfrac{q \cdot q^{2^{n-2}}}{q^{2^{n-2}}} = q \in S$ then s_{n-1} is conneted to s_0 and s_{n-2}. Hence, $d(s_i) \geq 2$, for all $i = 0, 1, \cdots, n-1$. Next is to show that $d(s_i) = 2$, for all $i = 0, 1, \cdots, n-1$. Indeed, for all $i = 1, 2, \cdots, n-3$, $\dfrac{q \cdot q^{2^{n-2}}}{q^{2^i}} = q^{2^{n-2} - 2^i + 1} \notin S$. Also for $\{s_i\}_{i=0}^{n-2}$, by Theorem 2.1, s_k and s_j are not connected if $|k - j| > 1$. Thus, $d(s_i) = 2$. Therefore, ϕ defines a labeling for a path. $\square$

Definition 2.3. A fan, denoted as F_n, with n vertices, is a graph with exactly 2 vertices of degree 2, $n-3$ vertices of degree 3, and 1 vertex of degree $n-1$.

Theorem 2.3. *Every fan graph F_n, $n \geq 3$ is a closed divisor graph.*

Proof. Let F_n be a fan with $n = 3$ and $S := \{s_i\}_{i=0}^{n-1}$.
But $F_3 \cong C_3$. So by Theorem 3.1.2, F_3 is a closed divisor graph.
Now, let $n \geq 4$ and $V(F_n) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, n-1$.

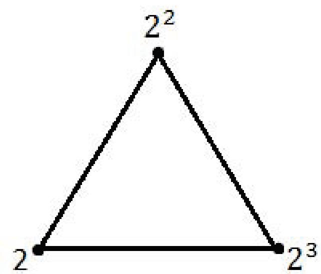

Figure 2: A closed divisor labeling of a cycle graph C_3.

Also, define a labeling $\phi : V(F_n) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} 1 & \text{if } i = 0; \\ q^{2^i} & \text{if } i = 1, 2, 3, \cdots, n-1 \text{ for any fixed } q \geq 2,\ q \text{ a positive integer;} \end{cases}$$

By the labeling ϕ defined above, it is clear that s_0 is connected to all s_i for all $i = 1, 2, \cdots, n-1$. From the proof of Theorem, 2.1, s_i and s_{i+1} are connected for all $i = 1, 2, \cdots, n-1$ and s_k,s_j are not connected for $|k-j| > 1$. Therefore, the labeling ϕ defines a fan. □

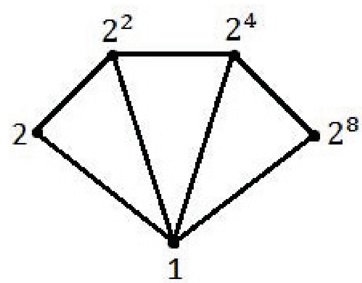

Figure 3: A closed divisor labeling of a fan graph F_5.

Definition 2.4. A wheel graph, denoted as W_n, with n vertices, is a graph with exactly one vertex of degree $n-1$, and $n-1$ vertices of degree 3.

Theorem 2.4. *Every wheel graph W_n, $n \geq 3$ is a closed divisor graph.*

Proof. Let W_n be a wheel graph with $n = 3$ vertices and $S := \{s_i\}_{i=0}^{n-1}$. But $W_3 \cong F_3$. So by Theorem 3.1.3, W_3 is a closed divisor graph.

Now, let $n \geq 4$ and $V(W_n) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, n-1$.

Also, define a labeling $\phi : V(W_n) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} 1 & \text{if } i = 0; \\ q^{2^{i-1}} & \text{if } i = 1, 2, 3, \cdots, n-2 \text{ for any fixed } q \geq 2,\ q \text{ a positive integer}; \\ q \cdot q^{2^{n-3}} & \text{if } i = n-1 \end{cases}$$

By the labeling ϕ defined above, it is clear that s_0 is connected to all $s_i, i = 1, 2, \cdots, n-1$. Also, by Theorem 2.2, $\{s_i\}_{i=1}^{n-1}$ form a cycle. Thus, $d(s_0) = n-1$ and $d(s_i) = 3, i = 1, 2, \cdots, n-1$. Therefore, the labeling ϕ defines a wheel. □

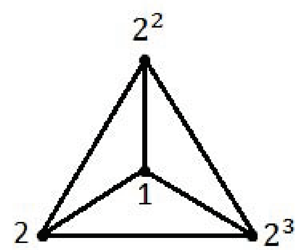

Figure 4: A closed divisor labeling of a wheel graph W_4.

Definition 2.5. A star, denoted as Sr_n, with n vertices, is a graph with exactly one vertex of degree $n-1$, and $n-1$ vertices of degree 1.

Theorem 2.5. *Every star graph Sr_n, $n \geq 3$ is a closed divisor graph.*

Proof. Let S_n be a star graph and $S := \{s_i\}_{i=0}^{n-1}$.

If $n = 3$, then $Sr_3 \cong P_3$; hence, by Theorem 3.1.1, Sr_3 is a closed divisor graph.

Now, let $n \geq 4$ and $V(Sr_n) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, n-1$.
Define a labeling $\phi : V(Sr_n) \rightarrow S$, where

$$\phi(v_i) = s_i = \begin{cases} 1 & \text{if } i = 0; \\ p_i & \text{if } i = 1, 2, 3, \cdots, n-1 \ p_i\text{'s are the first } i^{th} \text{ prime numbers;} \end{cases}$$

By the definition above, it clear that s_0 is connected to all $s_i, i = 1, 2 \cdots, n-1$. Also, since $s_i, i = 1, 2, \cdots, n-1$ are prime and distinct, it follows that they are all not connected. Hence, $d(s_0) = n-1$, and $d(s_1) = d(s_2) = d(s_3) = \cdots = d(s_{n-1}) = 1$. □

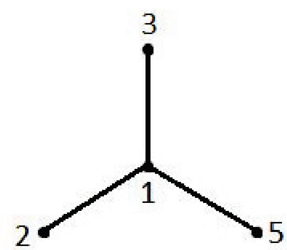

Figure 5: A closed divisor labeling of a star graph S_4.

Definition 2.6. A complete graph, denoted as K_n, with n vertices, is a graph where each vertex has a degree of $n-1$.

Theorem 2.6. *Every complete graph K_n, $n \geq 2$ is a closed divisor graph.*

Proof. Let K_n be a complete graph with $n = 2$ vertices and $S := \{s_i\}_{i=1}^{n}$.
But $K_2 \cong P_2$; hence, by Theorem 3.1.1, K_2 is a closed divisor graph.
Now, let $n \geq 3$ and $V(K_n) = \{v_1, v_2, \cdots, v_n\}, i = 0, 1, \cdots, n-1$.
Also, define a labeling $\phi : V(K_n) \rightarrow S$, where

$$\phi(v_i) = s_i = q^i, i = 1, 2, \cdots, n \text{ for any fixed } q \geq 2, \text{ q is a positive integer;}$$

Now, with the labeling ϕ, each s_k and $s_\ell, 1 \leq k \leq l \leq n-1$ are connected since $\dfrac{q^l}{q^k} = q^{l-k} \in S$. Therefore, $d(s_0) = d(s_1) = d(s_2) = \cdots = d(s_{n-1}) = n-1$. □

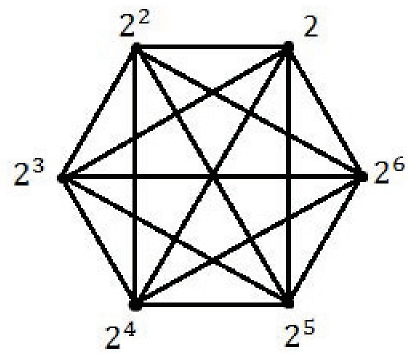

Figure 6: A closed divisor labeling of a complete graph K_6.

Definition 2.7. A complete bipartite graph, denoted as $K_{m,n}$, is a graph with two disjoint sets A and B where $|V(A)| = m$, and $|V(B)| = n$ such that no two vertices within the same set are adjacent. Also, each vertex of A is connected to each vertex of B; that is, each vertex of A is of degree n and each vertex of B is of degree m.

Theorem 2.7. *Every complete bipartite graph $K_{n+1,n}, n > 2$, is a closed divisor graph.*

Proof. Let $K_{n+1,n}$ be a complete bipartite graph with 2n+1 ≥ 5 vertices, n is a positive integer.

Let $V(K_{n+1,n}) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, 2n$ and $S := \{s_i\}_{i=0}^{2n}$.

Define the labeling $\phi : V(K_{m,n}) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} q^{1+3(\frac{i}{2})} & \text{if } i = 0, 2, 4, \cdots, 2n; \\ q^{2+3(\frac{i-1}{2})} & \text{if } i = 1, 3, 5, \cdots, 2n-1; \end{cases}$$

where q is any fixed positive integer greater than or equal to 2.

For $a \neq b$, $s_a, s_b \in \{s_0, s_2, s_4, \cdots, s_{2n}\}$ are not connected since $\dfrac{q^{1+3k}}{q^{1+3j}} = q^{3(j-k)} \notin S$. Similarly, for $a \neq b$, $s_a, s_b \in \{s_1, s_3, s_5, \cdots, s_{2n-1}\}$ are not connected since $\dfrac{q^{2+3k}}{q^{2+3j}} = q^{3(j-k)} \notin S$.

Let k be fixed. If $k > l$ then $\dfrac{q^{1+3k}}{q^{2+3l}} = q^{-1+3(k-l)} = q^{2+3(k-l-1)} \in S$. Also if $l \geq k$ then $\dfrac{q^{2+3l}}{q^{1+3k}} = q^{1+3(l-k)} \in S$. Thus, each $s_x, x = 0, 2, \cdots, 2n$ is connected to all $s_y, y = 1, 3, \cdots, 2n-1$. Therefore, we have disjoint sets X, Y where $S(X) = \{s_0, s_2, s_4, \cdots, s_{2n}\}$ and $S(Y) = \{s_1, s_3, s_5, \cdots, s_{2n-1}\}$.

Also, each s_x is of degree n and each s_y is of degree $n+1$. □

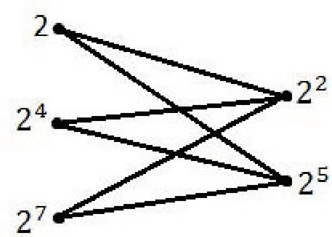

Figure 7: A closed divisor labeling of a complete bipartite graph $K_{3,2}$.

Theorem 2.8. *Every complete bipartite graph $K_{n,n}, n \geq 2$, is a closed divisor graph.*

Proof. Let $K_{n,n}$ be a complete bipartite graph with 2n $\geq$ 4 vertices, n is a positive integer.

Let $V(K_{n,n}) = \{v_0, v_1, v_2, \cdots, v_i\}, i = 0, 1, \cdots, 2n+1$ and $S := \{s_i\}_{i=0}^{2n+1}$.

Define the labeling $\phi : V(K_{n,n}) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} q^{1+3((\frac{i}{2}))} & \text{if } i = 0, 2, 4, \cdots, 2n; \\ q^{2+3(\frac{i-1}{2})} & \text{if } i = 1, 3, 5, \cdots, 2n+1; \end{cases}$$

By the labeling ϕ and same proof as done on Theorem 2.7, we have disjoint sets X, Y where $S(X) = \{s_0, s_2, s_4, \cdots, s_{2n}\}$ and $S(Y) = \{s_1, s_3, s_5, \cdots, s_{2n+1}\}$.

Also, each s_x and s_y are of degree n. □

Definition 2.8. A friendship graph, denoted as C_3^n, with $2n+1$ vertices and n-copies of C_3 is a graph with exactly $2n$ vertices of degree 2 and 1 vertex of degree $2n$.

Theorem 2.9. *Every friendship graph C_3^n, $n \geq 2$ is a closed divisor graph.*

Proof. Let C_3^n be a friendship graph with $n \geq 2$. Let $C_3^n = \{v_0, v_1, v_2, \cdots, v_i\}$, $i = 0, 1, \cdots, 2n$ and $S := \{s_i\}_{i=0}^{2n}$. We now define a labeling $\phi : V(C_3^n) \to S$, where

$$\phi(v_i) = s_i = \begin{cases} 1 & \text{if } i = 0; \\ p_i & \text{if } i = 1, 3, 5, \cdots, 2n-1,\ p_i\text{'s are the first } i^{th} \text{ prime numbers;} \\ s_{i-1}^2 & \text{if } i = 2, 4, 6, \cdots, 2n; \end{cases}$$

For $i = 1, 3, \cdots, 2n-1$, by Theorem 2.2, it is clear that $\{1, p_i, p_i^2\}$ forms a cycle. Also, it is clear that these cycles are connected by only one vertex $s_0 = 1$ since p_i's are prime. Hence, $d(s_0) = n - 1$ and $d(s_1) = d(s_3) = \cdots = d(s_{n-2}) = 2 = d(s_2) = d(s_4) = \cdots = d(s_{n-1})$. $\square$

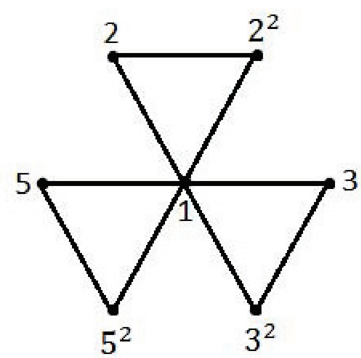

Figure 8: A closed divisor labeling of a friendship graph C_3^3.

Definition 2.9. A windmill graph, denoted as $W_n^{(m)}$, with $m(n-1)+1$ vertices is a graph formed by taking m-copies of a complete graph K_n with a vertex in common.

Theorem 2.10. *Every windmill graph $W_n^{(m)}$ is a closed divisor graph.*

Corollary. *If $n = 3$, then $W_3^{(m)}$ is isomorphic to the friendship graph C_3^m and thus a closed divisor graph.*

Proof. Consider $n \geq 3$ and $m \geq 2$.
Let $G_1, G_2, \cdots, G_m$ be complete graphs with n vertices and

$V(G_1) = \{v_{1,1}, v_{1,2}, v_{1,3} \cdots, v_{1,n}\}$
$V(G_2) = \{v_{2,1}, v_{2,2}, v_{2,3} \cdots, v_{2,n}\}$
$V(G_3) = \{v_{3,1}, v_{3,2}, v_{3,3} \cdots, v_{3,n}\}$

$$\vdots$$

$V(G_m) = \{v_{m,1}, v_{m,2}, v_{m,3} \cdots, v_{m,n}\}$
be the vertex sets of $G_1, G_2, G_3 \cdots, G_m$, respectively.

Given the vertex sets above, we can say that $V(W_n^{(m)}) = \bigcup_{i=1}^{m} V(G_i)$ where $v_{1,1} = v_{2,1}, = v_{3,1} = \cdots = v_{m,1} := v_0$. Now let $S := \{s_{j,i} : i = 1, 2, \cdots, n-1, j = 1, 2, \cdots, m\} \cup \{1\}$. Then define the labeling $\phi : V(W_n^{(m)}) \to S$, where $\phi(v_0) = 1$ and $\phi(v_{j,i}) = s_{j,i} = p_j^i, i = 1, 2, \cdots, n-1, j = 1, 2, \cdots, m$, for any fixed $p \geq 2$, p_m's are distinct positive prime numbers.
From Theorem 2.6, it is clear that for a fix j, $\{p_j^i\}_{i=1}^{n-1}$ form a complete graph. Also, since p_j's are prime and $\phi(v_0) = 1$ then all these complete graphs have only one common vertex, which is at v_0. □

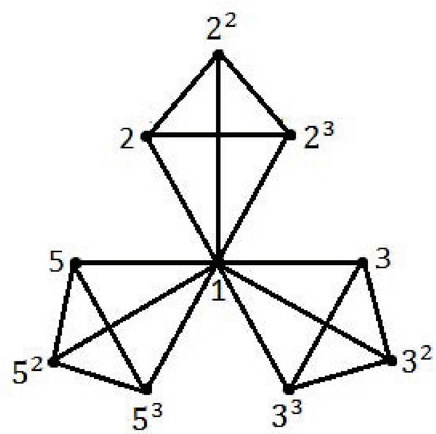

Figure 9: A closed divisor labeling of a windmill graph $W_{(4)}^3$.

Definition 2.10. The (m, n)-tadpole graph, denoted as $T_{(m,n)}$, with $m+n-1$ vertices is the graph obtained by joining a cycle graph C_m and a path graph P_n with a vertex in common.

Theorem 2.11. *Every (m, n)-tadpole graph is a closed divisor graph.*

Proof. Let C_m be a cycle with m vertices, where $m \geq 3$.
P_n be a path with n vertices, $m \geq 2$.

$V(C_m) = \{v_0, v_1, v_2, \cdots, v_{m-1}\}$ be the vertex set of C_m and;
$V(P_n) = \{= v_{m-1}, u_1, u_2, \cdots, u_{n-1}\}$ be the vertex set of P_n and $S := \{s_i\}_{i=0}^{n+m-2}$.

Given the vertex sets above, we can say that $V(T_{(m,n)}) = V(C_m) \bigcup V(P_n)$. Define the labeling $\phi : V(T_{(m,n)}) \to S$, where, for any fixed $q \geq 2$, $q \in \mathbb{Z}$,

$$\phi(v_i) = s_i = \begin{cases} q^{2^i} & \text{if } 0, 1, 2, \cdots, m-2; \\ q(q^{2^{m-2}}) & \text{if } i = m-1; \end{cases}$$

$$\phi(u_j) = s_{m-1+j} = (q(q^{2^{m-2}}))^{2^j}, j = 1, 2, \cdots, n-1$$

From Theorem 2.2, we know that $\{q^{2^i}\}_{i=0}^{m-2} \cup \{q(q^{2^{m-2}})\}$ forms a cycle. Also, from Theorem 2.1, $\{(q(q^{2^{m-2}}))^{2^j}\}_{j=1}^{n-1} \cup \{q(q^{2^{m-2}})\}$ forms a path. It is clear that $s_{m-1} = q(q^{2^{m-2}})$ is connecting these path and cycle. Next is to show that it is the only common vertex connecting these path and cycle. Indeed, since for $j > 0$ and $i = 0, 1, \cdots, m-3$, $\dfrac{(q(q^{2^{m-2}}))^{2^j}}{q^{2^i}} = q^{(2^{m-2}+1)2^j - 2^i} \notin S$. $\square$

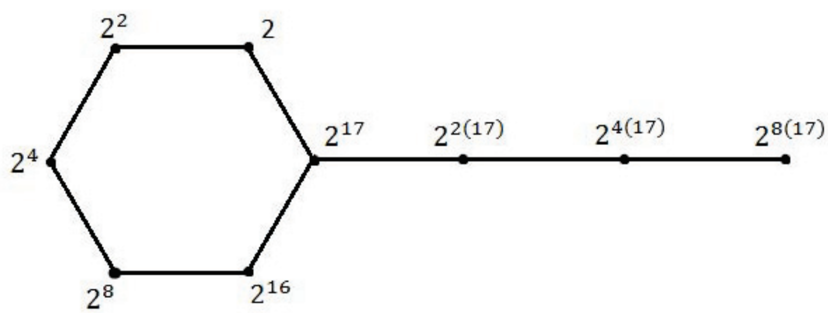

Figure 10: A closed divisor labeling of a tadpole graph $T_{(6,4)}$.

Definition 2.11. The (m, n)-lollipop graph, denoted as $L_{(m,n)}$, with $m(n-1)+1$ vertices is a graph obtained by joining a complete graph K_m and a path graph P_n with a common vertex.

Theorem 2.12. *Every (m, n)-lollipop graph $L_{(m,n)}$ is a closed divisor graph.*

Proof. Let K_m be a complete graph with m vertices, where $m \geq 3$.
P_n be a path graph with n vertices, $n \geq 2$.

$V(K_m) = \{v_0, v_1, v_2, \cdots, v_{m-1}\}$ be the vertex set of K_m and;
$V(P_n) = \{v_{m-1}, u_1, u_2, \cdots, u_{n-1}\}$ be the vertex set of P_n and $S := \{s_i\}_{i=0}^{n+m-2}$.

Define the labeling $\phi : V(L_{(m,n)}) \to S$, where for any fixed $q \geq 2$, $q \in \mathbb{Z}$,

$$\phi(v_i) = s_i = \begin{cases} q^{i+1} & \text{if } i = 0, 1, 2, \cdots, m-1; \end{cases}$$

$$\phi(u_j) = s_{m-1+j} = q^{2^j m}, j = 1, 2 \cdots, n-1$$

From Theorem 2.6, we know that $\{q^{i+1}\}_{i=0}^{m-1}$ forms a complete graph. Also, from Theorem 2.1, $\{(q^{2^j m}\}_{j=0}^{n-1}$ forms a path. It is clear that $s_{m-1} = q^m)$ is connecting these path and complete graph. Next is to show that it is the only common vertex connecting these path and cycle. Indeed, since for $j > 0$ and $i = 1, \cdots, m-1$, $\dfrac{q^{2^j m}}{q^i} = q^{2^j m - 2^i} \notin S$. □

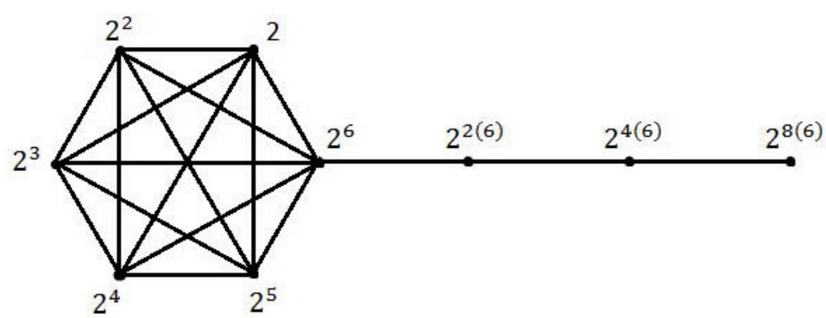

Figure 11: A closed divisor labeling of a lollipop graph $L_{(6,4)}$.

3 Proper Monographs as Closed Divisor Graphs

Let $S \subset \mathbb{R}^+$. A graph G is said to be a proper monograph if there exists a labeling $\omega : V(G) \to S$ such that two distinct vertices v_k, $v_l \in V(G)$ are adjacent if and only if $| \omega(v_k) - \omega(v_l) | \in S$.

Theorem 3.1. *If G is a proper monograph, then G is a closed divisor graph.*

Proof. Let G be a proper monograph. So there exists a bijective map $\phi : V(G) \to S$,where $v \in V(G) \longmapsto s \in S$, S is a set of positive integers such that two distinct vertices $v_j, v_k \in V(G)$ are connected if and only if $| s_j - s_k | \in S$.

Now, let $CD = \{q^{\phi(v)} : q \in \mathbb{Z}, q \geq 2, v \in V(G)\}$. Then consider the mapping $\omega : V(G) \to CD$ where $v \in V(G) \longmapsto q^{\phi(v)}$. It is clear that the mapping ω is bijective. So, let $u, v \in V(G)$ be connected vertices. Then $| \phi(u) - \phi(v) | \in S$ and $\phi(u), \phi(v) \in S$. Thus, by mapping ω defined above, $q^{\phi(u)}, q^{\phi(v)}$, and $q^{|\phi(u)-\phi(v)|} \in CD$; that is, $q^{\phi(u)} \mid q^{\phi(v)}$ or $q^{\phi(v)} \mid q^{\phi(u)}$. Therefore, the theorem holds. $\square$

Example: $S(P_n) = \{1, 2, 4, ..., 2^n\} \Rightarrow CD(P_n) = \{q^1, q^2, q^4, ..., q^{2^n}\}$
$S(C_n) = \{1, 2, 4, ..., 2^{n-1}, 1+2^{n-1}\} \Rightarrow CD(C_n) = \{q^1, q^2, q^4, ..., q^{2^{n-1}}, q^{1+2^{n-1}}\}$
$S(K_n) = \{1, 2, 3, ..., n\} \Rightarrow CD(K_n) = \{q^1, q^2, q^3, ..., q^n\}$
$S(K_{n,n-1}) = \{s : 1 \leq s \leq 3n-1, s \ncong 0(mod3)\} \Rightarrow CD(K_{n,n-1}) = \{q^s : 1 \leq s \leq 3n-1, s \ncong 0(mod3)\}$

Remark Conversely, with same proof on theorem 3.1. A closed divisor graph G with vertex labeling $f : V(G) \to CD$, where $f(v_i) = q^i, i = 1, 2, 3, ...n$, for any fixed $q \in Z^+$, is a proper monograph.

References

[1] Sugeng, K.A. , Ryan, J., *On several classes of monographs*, Aust. Journ. Comb. Vol. 37,277-284 (2007)

[2] Gallian, J.A., *A Dynamic Survey of Graph Labeling*, Elec. J. Combi. 9 (2014)

[3] Chartrand G., Muntean R., Saenpholphat V., Zhang P., *Which Graphs are Divisor Graphs?*,Congr. Numer., 151(2001) 189-200.

[4] Al-Ezeh, H., Abughneim, O. & Al-addasi, S., *Further new properties of divisor graphs*, Combin. Math. Combin. Comput. (2012)

[5] Fontanil, L.L., Panopio, R.G., *Independent set and vertex covering in a proper monograph determined through a signature*, Aust. J. Comb., Vol. 59(1), 64-71 (2012).

[6] Frayer, C., *Properties of Divisor Graphs*,Rose-Hulman Undergraduate Mathematics Journal: Vol. 4 : Iss. 2 , Article 4. (2003)