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UNIQUE COMMON FIXED POINT THEOREMS FOR SELF-MAPPINGS ON HILBERT SPACE

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Abstract

In this paper we have investigated a common fixed point for a pair and a sequence of self mappings over a closed subset of Hilbert space satisfying certain contraction inequalities involving rational square expressions as well as some positive powers of the terms. Finally these results are generalized for taking positive integers powers of the self mappings over the spaces which are generalizations of well-known results

Introduction

Essentially, fixed point theorems provide the conditions under which maps have solutions. First the existence and unique fixed point was given by the mathematician Banach in 1922, which was acclaimed as Banach contraction principle which has an important role in the development of various results connected with Fixed point Theory and Approximation Theory. Later this celebrated principle has been generalized by many authors Nadler [9], Sehgal [11], Wong [16], Jaggi [4, 5], Kannan [6], Fisher [2], Khare [7] etc. Also Ganguly and

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Bandyopadhya [3], Dass and Gupta [1], Koparde and Waghmode [8], Pandhare [10], Smart, [14], Veerapandi and Anil Kumar [15], Seshagiri Rao etl. [12, 13] have investigated the properties of fixed point mappings in complete metric spaces and as well as in Hilbert spaces. Motivated by the above results, the theorems which we found here are the analogous to the result of [1, 8, 10] involving a pair of mappings and a sequence of self mappings defined over a closed subset of a Hilbert space satisfying certain rational inequalities which are expressions of squares terms or some positive powers of the terms . Further these results are again extended for some positive integer powers of the self-mappings in the contractive inequalities. In all cases a unique common fixed point for a pair or a sequence of self-mappings is observed.

Main Results

Theorem 1 *Let T_1 and T_2 be two self-mappings of a closed subset X of a Hilbert space satisfying the inequality*

$$\| T_1x - T_2y \|^2 \leq \alpha \frac{\| x - T_2y \|^2 \; [1 + \| y - T_1x \|^2]}{1 + \| x - y \|^2} + \beta \| y - T_1x \|^2 + \gamma \| x - y \|^2$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are non-negative reals with $4(\alpha+\beta)+\gamma < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Let us start with an arbitrary point $x_0 \in X$. We define a sequence $\{x_n\}$ as

$$x_1 = Tx_0, \; x_2 = Tx_1, \; x_3 = Tx_2, \cdots$$

In general,

$$x_{2n+1} = T_1x_2n, \; x_{2n+2} = T_2x_{2n+1}, \; \text{for } n = 0, 1, 2, 3 \cdots$$

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence in X. For this we consider

$$\begin{aligned} \| x_{2n+1} - x_{2n} \|^2 &= \| T_1x_{2n} - T_2x_{2n-1} \|^2 \leq \\ &\leq \alpha \tfrac{\|x_{2n} - T_2x_{2n-1}\|^2 [1 + \|x_{2n-1} - T_1x_{2n}]}{1 + \|x_{2n} - x_{2n-1}\|^2} \\ &+ \beta \| x_{2n-1} - T_1x_{2n} \|^2 + \gamma \| x_2n - x_{2n-1} \|^2 \end{aligned}$$

It follows that $\| x_{2n+1} - x_{2n} \|^2 \leq k \| x_{2n} - x_{2n-1} \|^2$ where $k = \frac{2\beta+\gamma}{1-2\beta}$.

A similar calculation indicates that

$$\| x_{2n+2} - x_{2n+1} \|^2 \leq p(n) \| x_{2n+1} - x_{2n} \|^2$$

where $p(n) = \frac{(2\alpha+\gamma)+\gamma\|x_{2n+1}-x_{2n}\|^2}{(1-2\alpha)+\|x_{2n+1}-x_{2n}\|^2}$, since $4(\alpha+\beta)+\gamma<1$. We see that $k<1$ and $p(n)<1$ for all n.

Suppose that $P = \max\{p(n) : n = 1, 2, \cdots\}$ and $\lambda^2 = \max\{k, P\}$. Then, $0<\lambda<1$, as a result of which we get $\| x_{n+1} - x_n \| \leq \lambda \| x_n - x_{n-1} \|$. Repeating the above process in a similar manner, we get

$$\| x_{n+1} - x_n \| \leq \lambda^n \| x_1 - x_0 \|, \ n \geq 1.$$

Taking $n \to \infty$, we obtain $\| x_{n+1} - x_n \| \to 0$. It follows that the sequence $\{x_n\}$ is a Cauchy sequence. But X is a closed subset of Hilbert space and so by the completeness of X, there is some $\mu \in X$ such that

$$x_n \to \mu \text{ as } n \to \infty.$$

Consequently, the sequences $\{x_{2n+1}\} = \{T_1 x_{2n}\}$ and $\{x_{2n+2} = \{T_2 x_{2n+1}\}$ converge to the same limit μ. We now show that μ is a common fixed point of both T_1 and T_2. For this, in view of the hypothesis, note that

$$\| \mu - \ T_1\mu \|^2 = \| (\mu - x_{2n+2}) + x_{2n+2} - T_1\mu \|^2$$

$$\leq \| \mu - x_{2n+2} \|^2 + \alpha \frac{\|\mu - T_2 x_{2n+1}\|^2 [1 + \|x_{2n+1} - T_1\mu\|^2]}{1 + \|\mu - x_{2n+1}\|^2} + \beta \| x_{2n+1} - T_1\mu \|^2$$

$$+ \gamma \| x_{2n+2} - T_1\mu \|^2 + 2 \| \mu - x_{2n+2} \| \| x_{2n+2} - T_1\mu \|$$

Letting $n \to \infty$, we obtain $\| \mu - T_1\mu \|^2 \leq \beta \| \mu - T_1\mu \|^2$, since $0<\beta<1$. It follows that $T_1\mu = \mu$.

Similarly by making use of hypothesis we get $T_2\mu = \mu$ by considering the following

$$\| \mu - T_2\mu \|^2 = \| (\mu - x_{2n+1} + (x_{2n+1} - T_2\mu) \|^2$$

Finally, in order to prove the uniqueness of μ, suppose that μ and $\nu(\mu \neq \nu)$ are fixed points of T_1 and T_2. Then from the inequality, we obtain

$$\| \mu - \nu \|^2 \leq \alpha \frac{\| \mu - T_2\nu \|^2 \ [\| \nu - T_1\mu \|^2]}{1 + \| \mu - \nu \|^2} + \beta \| \nu - T_1\mu \|^2 + \gamma \| \nu - \mu \|^2$$

This inturn, implies that $\| \mu - \nu \|^2 \leq (\alpha+\beta+\gamma) \| \nu - \mu \|^2$. This gives a contradiction; for $\alpha+\beta+\gamma<1$. Thus, T_1 and T_2 have a unique common fixed point in X.

In the following theorem the numbers of terms are increased to two in the inequality of the above theorem.

Theorem 2. *Let X be a closed subset of a Hilbert space and T_1, T_2 be two mappings on X itself satisfying*

$$\| T_1^p x - T_2^q y \|^2 \leq \alpha \frac{\| x - T_2^q y \|^2 \ [1 + \| y - T_1^p x \|^2}{1 + \| x - y \|^2} + \beta \| y - T_1^p x \|^2 + \gamma \| x - y \|^2$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are non-negative reals with $4(\alpha+\beta)+\gamma < 1$ and p, q are positive integers. Then T_1 and T_2 have a unique common fixed point in X .

Proof. From Theorem 1, T_1^p and T_2^q have a unique common fixed point $\mu \in X$, that is $T_1^p\mu = \mu$ and $T_2^q\mu = \mu$.

From $T_1^p(T_1\mu) = T_1(T_1^p\mu) = T_1\mu$, it follows that $T_1\mu$ is a fixed point of T_1^p. But μ is a unique fixed point of T_1^p, we have $T_1\mu = \mu$. Similarly, we get $T_2\mu = \mu$. Thus, μ is a common fixed point of T_1 and T_2.

For uniqueness, let ν be another fixed point of T_1 and T_2, i.e., $T_1\nu = T_2\nu = \nu$. Then

$$\| \mu - \nu \|^2 = \| T_1^p\mu - T_2^q\nu \|^2 \leq \alpha \frac{\|\mu - T_2^q\nu\|^2 \, [1+\|\nu - T_1^p\mu\|^2]}{1+\|\mu-\nu\|^2} + \beta \| \nu - T_1^p\mu \|^2 + \gamma \| \mu - \nu \|^2 .$$

It would imply that $\| \mu - \nu \|^2 \leq (\alpha+\beta+\gamma) \| \mu - \nu \|^2$. Thus, $\mu = \nu$, since $\alpha + \beta + \gamma < 1$. Hence μ is a unique common fixed point of T_1 and T_2 in X, completing the proof of the theorem.

In the following theorem we have taken a sequence of mappings on a closed subset of a Hilbert space which converges point-wise to a limit mapping and show that if this limit mapping has a fixed point, then this fixed point is also the limit of fixed points of the mappings of the sequence.

Theorem 3. *Let X be a closed subset of a Hilbert space and let $\{T_i\}$ be a sequence of mappings from X into itself converging point-wise to T satisfying the following condition*

$$\| T_ix - T_iy \|^2 \leq \alpha \frac{\| x - T_iy \|^2 \; [1+ \| y - T_ix \|^2]}{1+ \| x-y \|^2} + \beta \| y - T_ix \|^2 + \gamma \| x - y \|^2$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are positive reals with $4(\alpha+\beta)+\gamma < 1$. If each T_i has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_i\}$ converges to μ.

Proof. Since μ_i is a fixed point of T_i, we have

$$\begin{aligned} \| \mu - \mu_n \|^2 &= \| T\mu - T_n\mu_n \|^2 \\ &= \| (T\mu - T_n\mu_n) + (T_n\mu - T_n\mu_n) \|^2 \\ &\leq \| T\mu - T_n\mu \|^2 + \| T_n\mu - T_n\mu_n \|^2 \\ &\qquad + 2 \| T\mu - T_n\mu \| \| T_n\mu - T_n\mu_n \| . \end{aligned}$$

This implies that

$$\| \mu-\mu_n \|^2 \leq \| T\mu-T_n\mu \|^2 +\alpha\frac{\| \mu-T_n\mu_n \|^2\ [1+ \| \mu_n-T_n\mu \|^2]}{1+ \| \mu-mu_n \|^2}$$
$$+\beta \| \mu_n-T_n\mu \|^2 +\gamma \| \mu-\mu_n \|^2 +2 \| T\mu-T_n\mu \| \| T_n\mu-T_n\mu_n \|$$

Letting $n\to\infty$, we get $T_n\mu\to T\mu,\ 2 \| T\mu-T_n\mu \| \| T_n\mu-T_n\mu_n \|\to 0$, and hence $\lim\limits_{n\to\infty} \| \mu-\mu_n \|^2\leq(\alpha+\beta+\gamma)\lim\limits_{n\to\infty} \| \mu-mu_n \|^2$. It impplies that $\lim\limits_{n\to\infty} \| \mu-\mu_n \|=0$, since $\alpha+\beta+\gamma<1$. Thus, $\mu_n\to\mu$ as $n\to\infty$, completing our proof. □

By following the above proofs, a pair of self mappings 1 T and 2 T defined over X have a common fixed point satisfying the below contraction equalities with some positive powers of the terms

Corollary 1 *The two self mappings T_1 and T_2 defined over a closed subset X of a Hilbert space satisfying the inequality*

$$\| T_1x-T_2y \|^r\leq\alpha\frac{\| x-T_2y \|^r\ \ [1+ \| y-T_1x \|^r]}{1+ \| x-y \|^r}+\beta \| y-T_1x \|^r +\gamma \| x-y \|^r$$

for all $x,y\in X,\ x\neq y$ and $r\in\mathbb{N}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, where α,β,γ are non-negative reals with $4(\alpha+\beta)+\gamma<1$. Then T_1 and T_2 have a unique common fixed point in X.

Corollary 2. *Let T_1 and T_2 be two self mappings over a closed subset X of a Hilbert space satisfying the contraction condition*

$$\| T_1^px-T_2^qy \|^r\leq\alpha\frac{\| x-T_2^qy \|^r\ \ [1+ \| y-T_1^px \|^r]}{1+ \| x-y \|^r}+\beta \| y-T_1^px \|^r +\gamma \| x-y \|^r$$

for all $x,y\in X,\ x\neq y$ and $r\in\mathbb{N}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, where α,β,γ are non-negative reals with $4(\alpha+\beta)+\gamma<1$ and p,q are positive integers. Then T_1 and T_2 have a unique common fixed point in X.

Corollary 3. *Let X be a closed subset of a Hilbert space and let $\{T_i\}$ be a sequence of mappings from X into itself converging pointwise to T satisfying the following condition*

$$\| T_ix-T_y \|^r\leq\alpha\frac{x-T_iy \|^r\ \ [1+ \| y-T_ix \|^r]}{1+ \| x-y \|^r}+\beta \| y-T_ix \|^r +\gamma \| x-y \|^r$$

for all $x,y\in X,\ x \not/ y$ and $r\in\mathbb{N}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, where α,β,γ are non-negative reals with $4(\alpha+\beta)+\gamma<1$. If each T_i has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_i\}$ converges to μ.

By refining Theorem 1, including two rational square terms in the contraction conditions we have the following theorem which will give a common unique fixed point in a closed subset of Hilbert space for two self mappings.

Theorem 4. *Let X be a closed subset of a Hilbert space and let T_1 and T_2 be two self mappings defined over it satisfying the following condition, then T_1 and T_2 have a unique common fixed point in X.*

$$\begin{aligned}\| T_1x - T_2y \|^2 \leq\ & \alpha\frac{\| x - T_2y \|^2\ \ [1+ \| y - T_1x \|^2]}{1+ \| x - y \|^2} \\ & + \beta\frac{\| y - T_1x \|^2\ \ [1+ \| x - T_2y \|^2]}{1+ \| x - y \|^2} + \gamma \| x - y \|^2\end{aligned}$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are non-negative reals with $4(\alpha+\beta)+\gamma < 1$.

Proof. Let us construct a sequence$\{x_n\}$ for an arbitrary point $x_0 \in X$ as follows

$$x_{2n+1} = T_1x_{2n},\ x_{2n+2} = T_2x_{2n+1},\ \text{for } n = 0, 1, 2, 3, \cdots$$

For examining the Cauchy sequence nature of $\{x_n\}$ in X, we consider

$$\begin{aligned}\| x_{2n+1} - x_{2n} \|^2 =& \| T_1x_{2n} - T_2x_{2n-1} \|^2 \\ \leq\ & \alpha\frac{\| x_{2n} - T_2x_{2n-1} \|^2\ \ [1+ \| x_{2n-1} - T_1x_{2n} \|^2]}{1+ \| x_{2n} - x_{2n-1} \|^2} \\ & + \beta\frac{\| x_{2n-1} - Tx_{2n} \|^2\ \ [1+ \| x_{2n} - T_2x_{2n-1} \|^2]}{1+ \| x_{2n} - x_{2n-1} \|^2} \\ & + \gamma \| x_{2n} - x_{2n-1} \|^2\end{aligned}$$

which implies that

$$\| x_{2n+1} - x_{2n} \|^2 \leq p(n) \| x_{2n} - x_{2n-1} \|^2,$$

where

$$p(n) = \frac{(2\beta + \gamma) + \gamma \| x_{2n} - x_{2n-1} \|^2}{(1 - 2\beta)+ \| x_{2n} - x_{2n-1} \|^2}.$$

Similarly, we get

$$\| x_{2n+2} - x_{2n+1} \|^2 \leq q(n) \| x_{2n+1} - x_{2n} \|^2,$$

where

$$q(n) = \frac{(2\alpha + \gamma) + \gamma \| x_{2n+1} - x_{2n} \|^2}{(1 - 2\alpha)+ \| x_{2n+1} - x_{2n} \|^2}.$$

Since $4(\alpha+\beta)+\gamma < 1$, $p(n) < 1$ and $q(n) < 1$ for all n, put $\lambda^2 = \max\{P, Q\}$ where $P = \max\{p(n) : n \in \mathbb{N}\}$, $Q = \max\{q(n) : n \in \mathbb{N}\}$. Then $0 < \lambda < 1$ and

$$\| x_{n+1} - x_n \| \leq \| x_n - x_{n-1}$$

Continuing the above process, we get $\| x_{n+1} - x_n \| \leq \lambda^n \| x_1 - x_0 \|$, $n \geq 1$. Taking $n \to \infty$, we obtain $\| x_{n+1} - x_n \| \to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in X and so it has a limit μ in X.

Since the sequences $\{x_{2n+1}\} = \{T_1 x_{2n}$ and $\{x_{2n+2}\} = \{T_2 x_{2n+1}\}$ are subsequences of $\{x_n\}$, they have the same limit μ. Next, we show that μ a is a common fixed point of T_1 and T_2. For this, using the inequality, we arrive at

$$\begin{aligned}\| \mu - T_1\mu \|^2 =& \| (\mu - x_{2n+2} + (x_{2n+2} - T_1\mu \|^2 \\ &\leq \| \mu - x_{2n+2} \|^2 + \alpha \frac{\| \mu - T_2 x_{2n+1} \|^2 \ [1 + \| x_{2n+1} - T_1\mu \|^2]}{1 + \| \mu - x_{2n+1} \|^2} \\ &+ \beta \frac{\| x_{2n+1} - T_2\mu \|^2 \ [1 + \| \mu - T_1 x_{2n+1} \|^2]}{1 + \| \mu - x_{2n+1} \|^2} \\ &+ \gamma \| \mu - x_{2n+1} \|^2 + 2 \| \mu - x_{2n+2} \| \, \| x_{2n+2} - T_1\mu \|\end{aligned}$$

Letting $n \to \infty$, we obtain $\| \mu - T_1\mu \|^2 \leq \beta \| \mu - T_1\mu \|^2$. Since $0 < \beta < 1$, it follows immediately that $T_1\mu = \mu$. Similarly, by using the inequality, we get $T_2\mu = \mu$ by considering the following

$$\begin{aligned}\| \mu - T_2\mu \|^2 =& \| (\mu - x_{2n+1}) + (x_{2n+1} - T_2\mu \|^2 \\ &\leq \| \mu - x_{2n+1} \|^2 + \alpha \frac{\| x_{2n} - T_2\mu_2 \|^2 \ [1 + \| \mu - T_1 x_{2n} \|^2]}{1 + \| x_{2n} - \mu \|^2} \\ &+ \beta \frac{\| \mu - T_1 x_{2n} \|^2 \ [1 + \| x_{2n} - T_2\mu \|^2}{1 + \| x_{2n} - \mu \|^2} \\ &+ \gamma \| x_{2n} - \mu \|^2 + 2 \| \mu - x_{2n+1} \| \, \| x_{2n+1} - T_2\mu \|\end{aligned}$$

Next, we want to show that μ is a unique fixed point of T_1, T_2. Let us suppose that ν $(\mu \neq \nu)$ is also a common fixed point of T_1 and T_2. Then, in view of hypothesis, we have

$$\begin{aligned}\| \mu - \nu \|^2 \leq& \frac{\| \mu - T_2\nu \|^2 \ [1 + \| \nu - T_2\mu \|^2]}{1 + \| \mu - \nu \|^2} \\ &+ \beta \frac{\| \nu - T_1\mu \|]^2 [1 + \| \mu - T_2\nu \|]^2}{1 + \| \mu - \nu \|^2} + \gamma \| \nu - \mu \|^2 .\end{aligned}$$

It would imply that

$$\| \mu - \nu \|^2 \leq (\alpha + \beta + \gamma) \| \nu - \mu \|^2 .$$

This is a contradiction, since $\alpha+\beta+\gamma<1$. It follows that $\mu=\nu$, and therefore the common fixed point is unique.

Similarly we can obtain a unique common fixed point for self mappings satisfying the following inequalities.

Corollary 4. *Let X be a closed subset of a Hilbert space and T_1, T_2 be two mappings from X to itself satisfying*

$$\| T_1^p x - T_2^q y \|^2 \leq \alpha \frac{\| x - T_2^q y \|^2 \, [1 + \| y - T_1^p x \|^2]}{1 + \| x - y \|^2} + \beta \frac{\| y - T_1^p x \|^2 \, [1 + \| x - T_2^q y \|^2]}{1 + \| x - y \|^2} + \gamma \| x - y \|^2$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are non-negative real numbers with $4(\alpha+\beta)+\gamma<1$ and p, q are positive integres. Then T_1 and T_2 have a unique common fixed point in X.

Corollary 5. *Let X be a closed subset of a Hilbert space and $\{T_i\}$ be a sequence of mappings from X into itself converging pointwise to T satisfying the following condition*

$$\| T_i x - T_i y \|^2 \leq \alpha \frac{\| x - T_i y \|^2 \, [1 + \| y - T_1 x \|^2]}{1 + \| x - y \|^2} + \beta \frac{\| y - T_i x \|^2 \, [1 + \| x - T_i y \|^2]}{1 + \| x - y \|^2} + \gamma \| x - y \|^2$$

for all $x, y \in X$ and $x \neq y$, where α, β, γ are real numbers with $4(\alpha+\beta)+\gamma<1$. If each T_j has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_i\}$ converges to μ.

Again by taking the positive power of the terms in the contraction inequalities we have the following corollaries admits a unique common fixed point.

Corollary 6. *Let X be a closed subset of a Hilbert space and T_1, T_2 be two mappings from X itself satisfying*

$$\| T_1 x - T_2 y \|^r \leq \alpha \frac{\| x - T_2 y \|^r \, [1 + \| y - T_1 x \|^r]}{1 + \| x - y \|^r} + \beta \frac{\| y - T_1 x \|^r \, [1 + \| x - T_2 y \|^r}{1 + \| x - y \|^r} + \gamma \| x - y \|^r$$

for all $x, y \in X$, $x \neq y$, and $r \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where α, β, γ are non-negative real numbers with $4(\alpha+\beta)+\gamma<1$. Then T_1 and T_2 have a unique common fixed point in X.

Corollary 7. *Let X be a closed subset of a Hilbert space and T_1, T_2 be two mappings from X to itself satisfying*

$$\| T_1^p x - T_2^q y \|^r \leq \alpha \frac{\| x - T_2^q y \|^r \ [1+ \| y - T_1^p x \|^r]}{1+ \| x - y \|^r} + \beta \frac{\| y - T_1 P x \|^r \ [1+ \| x - T_2^q y \|^r]}{1+ \| x - y \|^r} + \gamma \| x - y \|^r$$

for all $x, y \in X,\ x \neq y$ and $r \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where α, β, γ are non-negative real numbers with $4(\alpha + \beta) + \gamma < 1$ and p, q are positive integers. Then T_1 and T_2 have a unique common fixed point in X.

Corollary 8. *Let X be a closed subset of a Hilbert space and let $\{T_i\}$ be a sequence of mappings from X into itself converging pointwise to T satisfying the following condition*

$$\| T_i x - T_i y \|^r \leq \alpha \frac{\| x - T_i y \|^r \ [1+ \| y - T_i x \|^r]}{1+ \| x - y \|^r} + \beta \frac{\| y - T_i x \|^r \ [1+ \| x - T_i y \|^r]}{1+ \| x - y \|^r} + \gamma \| x - y \|^r$$

for all $x, y \in X,\ x \neq y$ and $r \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where α, β, γ are positive real numbers with $4(\alpha + \beta) + \gamma < 1$. If each T_i has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_i\}$ converges to μ.

Conclusions

In this paper we have discussed about a unique common fixed point for a pair and a sequence of self mappings over a closed subset of Hilbert space satisfying certain rational inequalities involving square terms. Further the results are also extended for taking some positive powers of the terms in contraction conditions as well as positive integer powers of the self mappings on the space.

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