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AN ADAPTIVE METHOD ON THE QUENCHING TIME OF A NONLINEAR PARABOLIC EQUATION WITH RESPECT TO THE NON-LINEAR SOURCE AND NEUMANN CONDITIONS

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Abstract

In this paper, we introduce a new adaptive method for computing the numerical solutions of a class of quenching parabolic equations which exhibit a solution with one singularity. The continuity of the quenching time is studied in this paper where we have considered a parabolic equation with variable reaction which quenches in a finite time. For this fact, we have estimated the quenching time and have proved that it is continuous as a function of the nonlinear source for the following boundary value problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -b(x)u^{-p}(x,t), \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $p > 0$, $u_0 \in C^1([0,1])$, $u_0^{'}(0) = 0$ and $u_0^{'}(1) = 0$. The potential $b(x) \in C^1((0,1))$, positive in $[0,1]$. We find some conditions under which

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the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar study has been also investigated taking a discrete form of the above problem. Finally, we give some numerical experiments to illustrate our analysis.

1 Introduction

Consider the following boundary value problem

$$u_t(x,t) - u_{xx}(x,t) = -b(x)u^{-p}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \tag{3}$$

where $p > 0$, $u_0 \in C^1([0,1])$, $u_0'(0) = 0$ and $u_0'(1) = 0$. The potential $b(x) \in C^1((0,1))$, positive in $[0,1]$ and $b_0 = \max_{x\in[0,1]} b(x)$.

Definition 1.1. *We say that the classical solution u of (1)-(3) quenches in a finite time if there exists a finite time T_q such that $u_{\min}(t) > 0$ for $t \in [0, T_q)$ but*

$$\lim_{t\to T_q} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{0\leq x\leq 1} u(x,t)$. The time T_q is called the quenching time of the solution u.

The theoretical study of solutions for semilinear parabolic equations which quench in a finite time has been the subject of investigations of many authors (see [2], [4]–[7], [11], [12], [16] and the references cited therein). Local in time existence of a classical solution has been proved and this solution is unique. In addition, it is shown that if the initial data at (3) satisfies $u_0''(x) - b(x)u_0^{-p}(x) \leq -Au_0^{-p}(x)$ in $[0,1]$ where $A \in (0,1]$, then the classical solution u of (1)–(3) quenches in a finite time T and we have the following estimates

$$\frac{\min_{0\leq x\leq 1}(u_0(x))^{p+1}}{p+1} \leq T \leq \frac{\min_{0\leq x\leq 1}(u_0(x))^{p+1}}{A(p+1)},$$

$$(A(p+1))^{\frac{1}{p+1}}(T-t)^{\frac{1}{p+1}} \leq u_{\min}(t) \leq ((B(p+1))^{\frac{1}{p+1}}(T-t)^{\frac{1}{p+1}} \quad \text{for} \quad t \in (0,T),$$

(see, for instance [4]–[6]).

In this paper, we are interested in the numerical study of the phenomenon of quenching. Under some assumptions, we show that the solution of a semidiscrete form of (1)–(3) quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time goes to the real one when the mesh size goes to zero. Similar results have been also given for a discrete form of (1)–(3). Our work was motived by the papers in [1], [3] and [15]. In [1] and [15], the authors have used semidiscrete and discrete forms for some parabolic equations to study the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). In [3], some schemes have been used to study the phenomenon of extinction (we say that a solution extincts in a finite time if it becomes zero after a finite time for equations without singularities). One may also consult the papers in [8]–[10], where the authors have studied theoretically the dependence with respect to the initial data of the blow-up time of nonlinear parabolic problems. Concerning the numerical study, one may find some results in [13], [14], [18], [19] where the authors have proposed some numerical schemes for computing the numerical solutions for parabolic problems which present a solution with one singularity.

This paper is organized as follows. In the next section, we give some results about the discrete maximum principle. In the third section, under some conditions, we prove that the solution of a semidiscrete form of (1)–(3) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. In the fifth section, we study the results of sections 3 and 4 taking a discrete form of (1)–(3). Finally, in the last section, we give some numerical results to illustrate our analysis.

2 Properties of a semidiscrete problem

In this section, we give some results about the discrete maximum principle. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer and let $h = \frac{1}{I}$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of the problem (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} - \delta^2 U_i(t) = -b_i U_i^{-p}(t), \quad 0 \leq i \leq I, \quad t \in (0, T_q^h), \tag{4}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{5}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Here $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$ where

$$\|U_h(t)\|_{\inf} = \min_{0 \leq i \leq I} U_i(t).$$

When the time T_q^h is finite, we say that the solution $U_h(t)$ of (4)–(5) quenches in a finite time and the time T_q^h is called the quenching time of the solution $U_h(t)$.
The following lemma is a semidiscrete form of the maximum principle.

Lemme 2.1. *Let $\alpha_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$ and let $V_h \in C^1([0,T), \mathbb{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t) V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0,T), \tag{6}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{7}$$

Then $V_i(t) \geq 0$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let T_0 be any quantity satisfying the inequality $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that

$$\alpha_i(t) - \lambda > 0 \quad \text{for} \quad 0 \leq i \leq I, \quad t \in [0, T_0].$$

Set $m = \min_{0 \leq t \leq T_0} \|Z_h(t)\|_{\inf}$. Since $Z_h(t)$ is a continuous vector on the compact $[0, T_0]$, there exist $i_0 \in \{0, ..., I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{8}$$

$$\delta^2 Z_{i_0}(t_0) \geq 0. \tag{9}$$

From (6), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0. \tag{10}$$

We deduce from (8)–(10) that $(\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. Therefore, $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete. $\square$

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemme 2.2. *Let*

$$f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R}). If$$

$$V_h, W_h \in C^1(0,T), \mathbb{R}^{I+1})$$

are such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + f(W_i(t), t),$$

$$0 \le i \le I, \quad t \in (0,T),$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I,$$

then $V_i(t) < W_i(t), \quad 0 \le i \le I, \quad t \in (0,T).$

Proof. Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t \in (0,T)$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. We see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0,$$

$$\delta^2 Z_{i_0}(t_0) \ge 0.$$

Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \le 0,$$

which contradicts the first strict inequality of the lemma and this ends the proof. $\square$

3 Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (4)–(5) quenches in a finite time and estimate its semidiscrete quenching time. We need the following result about the operator δ^2.

Lemme 3.1. *Let* $U_h \in \mathbb{R}^{I+1}$ *be such that* $U_h > 0$. *Then, we have*

$$\delta^2(U^{-p})_i \ge -pU_i^{-p-1}\delta^2 U_i, \quad 0 \le i \le I.$$

Proof. Applying Taylor's expansion, we find that

$$\delta^2(U^{-p})_i = -pU_i^{-p-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p+1)}{2h^2}\theta_i^{-p-2}$$
$$+(U_{i-1} - U_i)^2 \frac{p(p+1)}{2h^2}\eta_i^{-p-2}, \quad 0 \le i \le I,$$

where θ_i is an intermediate value between U_i and U_{i+1}, η_i the one between U_{i-1} and U_i, $U_{-1} = U_1$, $U_{I+1} = U_{I-1}$, $\eta_0 = \theta_0$, $\eta_I = \theta_I$. Use the fact that $U_h > 0$ to complete the rest of the proof. □

The statement of the result about solutions which quench in a finite time is the following.

Theorem 3.1. *Let U_h be the solution of (4)–(5) and assume that there exists a positive constant A such that $b_i \ge A$ with $A \in (0,1]$ and the initial data at (5) satisfies*

$$\delta^2\varphi_i - b_i\varphi_i^{-p} \le -A\varphi_i^{-p}, \quad 0 \le i \le I. \tag{11}$$

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h \le \frac{\|\varphi_h\|_{\inf}^{p+1}}{A(p+1)}.$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + AU_i^{-p}(t), \quad 0 \le i \le I.$$

A straightforward calculation gives

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - ApU_i^{-p-1}\frac{dU_i}{dt} - A\delta^2(U^{-p})_i, \quad 0 \le i \le I.$$

From Lemma 3.1, we have $\delta^2(U^{-p})_i \ge -pU_i^{-p-1}\delta^2 U_i$, which implies that

$$\frac{dJ_i}{dt} - \delta^2 J_i \le \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - ApU_i^{-p-1}(\frac{dU_i}{dt} - \delta^2 U_i), \quad 0 \le i \le I.$$

Using (4), we arrive at

$$\frac{dJ_i}{dt} - \delta^2 J_i \le pb_iU_i^{-p-1}J_i, \quad 0 \le i \le I, \quad t \in (0, T_q^h).$$

From (11), we observe that $J_h(0) \leq 0$. We deduce from Lemma 2.1 that $J_h(t) \leq 0$ for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_i(t)}{dt} \leq -AU_i^{-p}(t), \quad 0 \leq i \leq I, \quad t \in (0, T_q^h). \tag{12}$$

These estimates may be rewritten in the following form $U_i^p dU_i \leq -Adt$, $0 \leq i \leq I$. Integrating the above inequalities over the interval (t, T_q^h), we get

$$T_q^h - t \leq \frac{(U_i(t))^{p+1}}{A(p+1)}, \quad 0 \leq i \leq I. \tag{13}$$

Using the fact that $\|\varphi_h\|_{\inf} = U_{i_0}(0)$ for a certain $i_0 \in \{0, ..., I\}$ and taking $t = 0$ in (13), we obtain the desired result. □

Remark 3.1. *The inequalities (13) imply that*

$$T_q^h - t_0 \leq \frac{\|U_h(t_0)\|_{\inf}^{p+1}}{A(p+1)} \quad for \quad t_0 \in (0, T_q^h),$$

and

$$\|U_h(t)\|_{\inf} \geq (A(p+1))^{\frac{1}{p+1}} (T_q^h - t)^{\frac{1}{1+p}} \quad for \quad t \in (0, T_q^h).$$

Remark 3.2. *Let U_h be the solution of (4)–(5). Then, we have*

$$T_q^h \geq \frac{\|\varphi_h\|_{\inf}^{p+1}}{B(p+1)},$$

and

$$\|U_h(t)\|_{\inf} \leq (B(p+1))^{\frac{1}{p+1}} (T_q^h - t)^{\frac{1}{p+1}} \quad for \quad t \in (0, T_q^h).$$

To prove these estimates, we proceed as follows. Introduce the function $v(t)$ defined as follows $v(t) = \|U_h(t)\|_{\inf}$ for $t \in [0, T_q^h)$. Let $t_1, t_2 \in [0, T_q^h)$. Then, there exist $i_1, i_2 \in \{0, ..., I\}$ such that $v(t_1) = U_{i_1}(t_1)$ and $v(t_2) = U_{i_2}(t_2)$. We observe that

$$v(t_2) - v(t_1) \geq U_{i_2}(t_2) - U_{i_2}(t_1) = (t_2 - t_1)\frac{dU_{i_2}(t_2)}{dt} + o(t_2 - t_1),$$

$$v(t_2) - v(t_1) \leq U_{i_1}(t_2) - U_{i_1}(t_1) = (t_2 - t_1)\frac{dU_{i_1}(t_1)}{dt} + o(t_2 - t_1),$$

which implies that $v(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} \geq \frac{dU_{i_2}(t_2)}{dt} + o(1) = \delta^2 U_{i_2}(t_2) - b_{i_2} U_{i_2}^{-p}(t_2) + o(1).$$

Obviously, $\delta^2 U_{i_2}(t_2) \geq 0$*. Letting* $t_1 \to t_2$*, we obtain* $\frac{dv(t)}{dt} \geq -Bv^{-p}(t)$ *for a.e.* $t \in (0, T_q^h)$ *or equivalently* $v^p dv \geq -Bdt$ *for a.e.* $t \in (0, T_q^h)$*. Integrate the above inequality over* (t, T_q^h) *to obtain* $T_q^h - t \geq \frac{(v(t))^{p+1}}{B(p+1)}$*. Since* $v(t) = \|U_h(t)\|_{\inf}$*, we arrive at* $T_q^h - t \geq \frac{\|U_h(t)\|_{\inf}^{p+1}}{B(p+1)}$ *and the second estimate follows. To obtain the first one, it suffices to replace* t *by* 0 *in the above inequality and use the fact that* $\|\varphi_h\|_{\inf} = \|U_h(0)\|_{\inf}$.

Remark 3.3. *If* $\varphi_i = \alpha$, $0 \leq i \leq I$*, where* α *is a positive constant, then one may take* $A = 1$*. It may imply that the potential equals to 1. In this case,*

$$T_q^h = \frac{\alpha^{p+1}}{p+1} \quad \text{and} \quad \|U_h(t)\|_{\inf} = (p+1)^{\frac{1}{p+1}}(T_q^h - t)^{\frac{1}{p+1}} \quad \text{for} \quad t \in (0, T_q^h).$$

4 Convergence of the semidiscrete quenching time

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and its semidiscrete quenching time converges to the real one when the mesh size goes to zero.
We denote

$$u_h(t) = (u(x_0,t), ..., u(x_I,t))^T \quad \text{and} \quad \|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|.$$

In order to obtain the convergence of the semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1. *Assume that the problem (1)-(3) has a solution* $u \in C^{4,1}([0,1] \times [0,T])$ *such that* $\min_{t\in[0,T]} u_{\min}(t) = \varrho > 0$ *and the initial data at (5) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{14}$$

Then, for h sufficiently small, the problem (4)–(5) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ *such that the following relation holds*

$$\max_{0\leq t\leq T} \|U_h(t) - u_h(t)\|_\infty = 0(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0. \tag{15}$$

Proof. Let $K > 0$ and $L > 0$ be such that

$$\frac{\|u_{xxxx}\|_\infty}{12} \leq K \quad \text{and} \quad pb_0(\frac{\rho}{2})^{-p-1} = L. \tag{16}$$

The problem (4)–(5) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h) \leq \min\{T, T_q^h\}$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{17}$$

The relation (14) implies that $t(h) > 0$ for h sufficiently small. By the triangle inequality, we obtain

$$\|U_h(t)\|_{\inf} \geq \|u_h(t)\|_{\inf} - \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t(h)),$$

which implies that

$$\|U_h(t)\|_{\inf} \geq \varrho - \frac{\varrho}{2} = \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{18}$$

Since $u \in C^{4,1}$, taking the derivative in x on both sides of (1) and due to the fact that u_x, u_{xt} vanish at $x = 0$ and $x = 1$, we observe that u_{xxx} also vanishes at $x = 0$ and $x = 1$. Applying Taylor's expansion, we discover that

$$u_{xx}(x_i, t) = \delta^2 u(x_i, t) - \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t), \quad 0 \leq i \leq I, \quad t \in (0, t(h)).$$

To establish the above equalities for $i = 0$ and $i = I$, we have used the fact that u_x and u_{xxx} vanish at $x = 0$ and $x = 1$. Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. From the mean value theorem, we have

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = b_0 p \theta_i^{-p-1} e_i + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t), \quad 0 \leq i \leq I, \ t \in (0, t(h)),$$

where θ_i is an intermediate value between $U_i(t)$ and $u(x_i, t)$. Using (16), (18), we arrive at

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq L|e_i(t)| + Kh^2, \quad 0 \leq i \leq I, \quad t \in (0, t(h)). \tag{19}$$

Introduce the vector $z_h(t)$ defined as follows

$$z_i(t) = e^{(L+1)t}(\|\varphi_h - u_h(0)\|_\infty + Kh^2), \quad 0 \leq i \leq I, \ t \in (0, t(h)). \tag{20}$$

A straightforward computation reveals that

$$\frac{dz_i}{dt} - \delta^2 z_i > L|z_i| + Kh^2, \quad 0 \leq i \leq I, \quad t \in (0, t(h)),$$

$$z_i(0) > e_i(0), \quad 0 \leq i \leq I.$$

It follows from Comparison Lemma 2.2 that

$$z_i(t) > e_i(t) \quad \text{for} \quad t \in (0, t(h)), \quad 0 \leq i \leq I.$$

By the same way, we also prove that

$$z_i(t) > -e_i(t) \quad \text{for} \quad t \in (0, t(h)), \quad 0 \leq i \leq I,$$

which implies that

$$\|U_h(t)-u_h(t)\|_\infty \leq e^{(L+1)t}(\|\varphi_h-u_h(0)\|_\infty+Kh^2) \quad \text{for} \quad t\in(0,t(h)).$$

Let us show that $t(h)=\min\{T,T_q^h\}$. Suppose that $t(h)<\min\{T,T_q^h\}$. From (17), we obtain

$$\frac{\varrho}{2}\leq \|U_h(t(h))-u_h(t(h))\|_\infty \leq e^{(L+1)T}(\|\varphi_h-u_h(0)\|_\infty+Kh^2).$$

Let us notice that both last formulas for $t(h)$ are valid for sufficiently small h. Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\varrho}{2}\leq 0$, which is impossible. Consequently $t(h)=\min\{T,T_q^h\}$.
Now, let us show that $t(h)=T$. Suppose that $t(h)=T_q^h<T$. Reasoning as above, we prove that we have a contradiction and the proof is complete. □

Now, we are in a position to prove the main theorem of this section.

Theorem 4.2. *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time T_q such that $u\in C^{4,1}([0,1]\times[0,T_q))$ and the initial data at (5) satisfies the condition (14). Under the hypothesis of Theorem 3.1, the problem (4)–(5) has a solution U_h which quenches in a finite time T_q^h and we have*

$$\lim_{h\to 0} T_q^h = T_q.$$

Proof. Let $0<\varepsilon<T_q/2$. There exists $\varrho\in(0,1)$ such that

$$\frac{1}{A}\frac{\varrho^{p+1}}{(p+1)}\leq\frac{\varepsilon}{2}. \tag{21}$$

Since u quenches in a finite time T_q, there exist $h_0(\varepsilon)>0$ and a time $T_0\in(T_q-\frac{\varepsilon}{2},T_q)$ such that $0<u_{\min}(t)<\frac{\varrho}{2}$ for $t\in[T_0,T_q)$, $h\leq h_0(\varepsilon)$. It is not hard to see that $u_{\min}(t)>0$ for $t\in[0,T_0]$, $h\leq h_0(\varepsilon)$. From Theorem 4.1, the problem (4)–(5) has a solution $U_h(t)$ and we get $\|U_h(t)-u_h(t)\|_\infty\leq\frac{\varrho}{2}$ for $t\in[0,T_0]$, $h\leq h_0(\varepsilon)$, which implies that $\|U_h(T_0)-u_h(T_0)\|_\infty\leq\frac{\varrho}{2}$ for $h\leq h_0(\varepsilon)$. Applying the triangle inequality, we find that

$$\|U_h(T_0)\|_{\inf}\leq\|U_h(T_0)-u_h(T_0)\|_\infty+\|u_h(T_0)\|_{\inf}\leq\frac{\varrho}{2}+\frac{\varrho}{2}=\varrho \quad \text{for} \quad h\leq h_0(\varepsilon).$$

From Theorem 3.1, $U_h(t)$ quenches at the time T_q^h. We deduce from Remark 3.1 and (21) that for $h\leq h_0(\varepsilon)$,

$$|T_q^h-T_q|\leq|T_q^h-T_0|+|T_0-T_q|\leq\frac{1}{A}\frac{\|U_h(T_0)\|_{\inf}^{p+1}}{(p+1)}+\frac{\varepsilon}{2}\leq\varepsilon,$$

which leads us to the desired result. □

5 Full discretizations

In this section, we study the phenomenon of quenching using a full discrete explicit scheme of (1)–(3). Approximate the solution $u(x,t)$ of the problem (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - b_i (U_i^{(n)})^{-p}, \quad 0 \le i \le I, \tag{22}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \le i \le I, \tag{23}$$

where $n \ge 0$,

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}.$$

If $U_h^{(n)} > 0$, then $-(U_i^{(n)})^{-p-1} \ge -\|U_h^{(n)}\|_{\inf}^{-p-1}$, $0 \le i \le I$, and a straightforward computation reveals that

$$U_0^{(n+1)} \ge \frac{2\Delta t_n}{h^2} U_1^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - b_i \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1}) U_0^{(n)},$$

$$U_i^{(n+1)} \ge \frac{\Delta t_n}{h^2} U_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - b_i \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1}) U_i^{(n)} + \frac{\Delta t_n}{h^2} U_{i-1}^{(n)}, \ 1 \le i \le I-1,$$

$$U_I^{(n+1)} \ge \frac{2\Delta t_n}{h^2} U_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - b_i \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1}) U_I^{(n)}.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T_q, we need to adapt the size of the time step so that we choose

$$\Delta t_n = \min\Big\{ \frac{(1-\tau)h^2}{2}, \tau \|U_h^{(n)}\|_{\inf}^{p+1} \Big\}$$

with $0 < \tau < 1$. We observe that $1 - 2\frac{\Delta t_n}{h^2} - b_i \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1} \ge 0$, which implies that $U_h^{(n+1)} > 0$. Thus, since by hypothesis $U_h^{(0)} = \varphi_h > 0$, if we take Δt_n as defined above, then using a recursion argument, we see that the positivity of the discrete solution is guaranteed. Here, τ is a parameter which will be chosen later to allow the discrete solution $U_h^{(n)}$ to satisfy certain properties useful to get the convergence of the numerical quenching time defined below.

If necessary, we may take $\Delta t_n = \min\{\frac{(1-\tau)h^2}{K}, \tau \|U_h^{(n)}\|_{\inf}^{p+1}\}$ with $K > 2$ because in this case, the positivity of the discrete solution is also guaranteed. The following lemma is a discrete form of the maximum principle.

Lemme 5.1. *Let $a_h^{(n)}$ and $V_h^{(n)}$ be two sequences such that $a_h^{(n)}$ is bounded and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \geq 0, \quad 0 \leq i \leq I, \quad n \geq 0, \tag{24}$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \tag{25}$$

Then $V_i^{(n)} \geq 0$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

Proof. If $V_h^{(n)} \geq 0$, then a routine computation yields

$$V_0^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_1^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty) V_0^{(n)},$$

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)}, \ 1 \leq i \leq I-1,$$

$$V_I^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty) V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$, we see that $1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty$ is nonnegative. From (25), we deduce by induction that $V_h^{(n)} \geq 0$ which ends the proof. $\square$

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemme 5.2. *Let $V_h^{(n)}$, $W_h^{(n)}$ and $a_h^{(n)}$ be three sequences such that $a_h^{(n)}$ is bounded and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)},$$

$$0 \leq i \leq I, \quad n \geq 0,$$

$$V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.$$

Then $V_i^{(n)} \leq W_i^{(n)}$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

Now, let us give a property of the operator δ_t stated in the following lemma. Its proof is quite similar to that of Lemma 3.1, so we omit it here.

Lemme 5.3. *Let $U^{(n)} \in \mathbb{R}$ be such that $U^{(n)} > 0$ for $n \geq 0$. Then we have*

$$\delta_t (U^{(n)})^{-p} \geq -p (U^{(n)})^{-p-1} \delta_t U^{(n)}, \quad n \geq 0.$$

The theorem below is the discrete version of Theorem 4.1.

Theorem 5.1. *Suppose that the problem (1)–(3) has a solution* $u \in C^{4,2}([0,1]\times[0,T])$ *such that* $\min_{t\in[0,T]} u_{\min}(t) = \rho > 0$. *Assume that the initial data at (23) satisfies the condition* (14). *Then, the problem (22)–(23) has a solution* $U_h^{(n)}$ *for* h *sufficiently small,* $0 \le n \le J$ *and the following relation holds*

$$\max_{0\le n\le J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0,$$

where J *is any quantity satisfying the inequality* $\sum_{n=0}^{J-1} \Delta t_n \le T$ *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each h, the problem (22)–(23) has a solution $U_h^{(n)}$. Let $N \le J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad n < N. \tag{26}$$

We know that $N \ge 1$ because of (14). Applying the triangle inequality, we have

$$\|U_h^{(n)}\|_{\inf} \ge \|u_h(t_n)\|_{\inf} - \|U_h^{(n)} - u_h(t_n)\|_\infty \ge \frac{\rho}{2} \quad \text{for} \quad n < N. \tag{27}$$

As in the proof of Theorem 4.1, using Taylor's expansion, we find that for $n < N$, $0 \le i \le I$,

$$\delta_t u(x_i, t_n) - \delta^2 u(x_i, t_n) + u^{-p}(x_i, t_n) = -\frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i, \widetilde{t}_n).$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get for $n < N$, $0 \le i \le I$,

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = b_0 p(\xi_i^{(n)})^{-p-1} e_i^{(n)} + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t_n) - \frac{\Delta t_n}{2} u_{tt}(x_i, \widetilde{t}_n),$$

where $\xi_i^{(n)}$ is an intermediate value between $u(x_i, t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, then there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le p b_0 (\xi_i^{(n)})^{-p-1} e_i^{(n)} + M h^2, \ 0 \le i \le I, \ n < N. \tag{28}$$

Set $L = p b_0 (\frac{\rho}{2})^{-p-1}$ and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + M h^2), \quad 0 \le i \le I, \ n < N.$$

A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} > pb_0(\xi_i^{(n)})^{-p-1} V_i^{(n)} + Mh^2,\ 0 \leq i \leq I,\ n < N, \tag{29}$$

$$V_i^{(0)} > e_i^{(0)}, \quad 0 \leq i \leq I. \tag{30}$$

We observe from (27) that $pb_0(\xi_i^{(n)})^{-p-1}$ is bounded from above by L. It follows from Comparison Lemma 5.2 that $V_h^{(n)} \geq e_h^{(n)}$. By the same way, we also prove that $V_h^{(n)} \geq -e_h^{(n)}$, which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \leq e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2),\ n < N. \tag{31}$$

Let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (31) and use (26), we find that

$$\frac{\rho}{2} \leq \|U_h^{(N)} - u_h(t_N)\|_\infty \leq e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh^2).$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is a contradiction and the proof is complete. $\square$

To handle the phenomenon of quenching for discrete equations, we need the following definition.

Definition 5.1. *We say that the solution $U_h^{(n)}$ of (22)-(23) quenches in a finite time if $\|U_h^{(n)}\|_{\inf} > 0$ for $n \geq 0$, but*

$$\lim_{n\to+\infty} \|U_h^{(n)}\|_{\inf} = 0 \quad \textit{and} \quad T_h^{\Delta t} = \lim_{n\to+\infty} \sum_{i=0}^{n-1} \Delta t_i < +\infty.$$

The number $T_h^{\Delta t}$ is called the numerical quenching time of $U_h^{(n)}$.

The following theorem reveals that the discrete solution $U_h^{(n)}$ of (22)-(23) quenches in a finite time under some hypotheses.

Theorem 5.2. *Let $U_h^{(n)}$ be the solution of (22)-(23). Suppose that there exists a constant $A \in (0,1]$ such that the initial data at (23) satisfies*

$$\delta^2\varphi_i - b_i\varphi_i^{-p} \leq -A\varphi_i^{-p}, \quad 0 \leq i \leq I. \tag{32}$$

Then $U_h^{(n)}$ is nonincreasing and quenches in a finite time $T_h^{\Delta t}$ which satisfies the following estimate

$$T_h^{\Delta t} \leq \frac{\tau\|\varphi_h\|_{\inf}^{p+1}}{1-(1-\tau')^{p+1}},$$

where $\tau' = A\min\{\frac{(1-\tau)h^2\|\varphi_h\|_{\inf}^{-p-1}}{2}, \tau\}$.

Proof. Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t U_i^{(n)} + A(U_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

A straightforward computation yields for $0 \leq i \leq I$, $n \geq 0$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t \left(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}\right) + A\delta_t (U_i^{(n)})^{-p} - A\delta^2 (U_i^{(n)})^{-p}.$$

Using (22), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = -(b_i - A)\delta_t (U_i^{(n)})^{-p} - A\delta^2 (U_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

It follows from Lemmas 5.3 and 3.1 that for $0 \leq i \leq I$, $n \geq 0$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \quad \leq \quad (b_i - A)p(U_i^{(n)})^{-p-1}\delta_t U_i^{(n)} + Ap(U_i^{(n)})^{-p-1}\delta^2 U_i^{(n)}.$$

We deduce from (22) that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \leq pb_i (U_i^{(n)})^{-p-1} J_i^{(n)}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

Obviously, the inequalities (32) ensure that $J_h^{(0)} \leq 0$. Applying Lemma 5.1, we get $J_h^{(n)} \leq 0$ for $n \geq 0$, which implies that

$$U_i^{(n+1)} \leq U_i^{(n)}(1 - A\Delta t_n (U_i^{(n)})^{-p-1}), \quad 0 \leq i \leq I, \quad n \geq 0. \tag{33}$$

These estimates reveal that the sequence $U_h^{(n)}$ is nonincreasing. By induction, we obtain $U_h^{(n)} \leq U_h^{(0)} = \varphi_h$. Thus, the following holds

$$A\Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1} \geq A \min\{\frac{(1-\tau)h^2\|\varphi_h\|_{\inf}^{-p-1}}{2}, \tau\} = \tau'. \tag{34}$$

Let i_0 be such that $\|U_h^{(n)}\|_{\inf} = U_{i_0}^{(n)}$. Replacing i by i_0 in (33), we obtain

$$\|U_h^{(n+1)}\|_{\inf} \leq \|U_h^{(n)}\|_{\inf}(1 - \tau'), \quad n \geq 0, \tag{35}$$

and by iteration, we arrive at

$$\|U_h^{(n)}\|_{\inf} \leq \|U_h^{(0)}\|_{\inf}(1 - \tau')^n = \|\varphi_h\|_{\inf}(1 - \tau')^n, \quad n \geq 0. \tag{36}$$

Since the term on the right hand side of the above equality goes to zero as n approaches infinity, we conclude that $\|U_h^{(n)}\|_{\inf}$ tends to zero as n approaches infinity. Now, let us estimate the numerical quenching time. Due to (36) and the restriction $\Delta t_n \leq \tau \|U_h^{(n)}\|_{\inf}^{p+1}$, it is not hard to see that

$$\Sigma_{n=0}^{+\infty} \Delta t_n \leq \Sigma_{n=0}^{+\infty} \tau \|\varphi_h\|_{\inf}^{p+1} [(1 - \tau')^{p+1}]^n.$$

Use the fact that the series on the right hand side of the above inequality converges towards $\frac{\tau\|\varphi_h\|_{\inf}^{p+1}}{1-(1-\tau')^{p+1}}$ to complete the rest of the proof. $\square$

Remark 5.1. *From (35), we deduce by induction that*

$$\|U_h^{(n)}\|_{\inf} \leq \|U_h^{(q)}\|_{\inf}(1-\tau')^{n-q} \quad \text{for} \quad n \geq q,$$

and we see that

$$T_h^{\Delta t} - t_q = \Sigma_{n=q}^{+\infty}\Delta t_n \leq \Sigma_{n=q}^{+\infty}\tau\|U_h^{(q)}\|_{\inf}^{p+1}[(1-\tau')^{p+1}]^{n-q},$$

which implies that

$$T_h^{\Delta t} - t_q \leq \frac{\tau\|U_h^{(q)}\|_{\inf}^{p+1}}{1-(1-\tau')^{p+1}}.$$

Since $\tau^{'} = A\min\{\frac{(1-\tau)h^2\|\varphi_h\|_{\inf}^{-p-1}}{2}, \tau\}$, *if we take* $\tau = h^2$, *we get*

$$\frac{\tau^{'}}{\tau} = A\min\{\frac{(1-h^2)\|\varphi_h\|_{\inf}^{-p-1}}{2}, 1\} \geq A\min\{\frac{\|\varphi_h\|_{\inf}^{-p-1}}{4}, 1\}.$$

Therefore, there exist constants c_0, c_1 *such that*
$0 \leq c_0 \leq \tau/\tau^{'} \leq c_1$ *and*

$$\tau/(1-(1-\tau^{'})^{p+1}) = O(1),$$

for the choice $\tau = h^2$.

In the sequel, we take $\tau = h^2$.
Now, we are in a position
to state the main theorem of this section.

Theorem 5.3. *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time* T_q *and* $u \in C^{4,2}([0,1]\times[0,T_q))$. *Assume that the initial data at (23) satisfies the condition (14). Under the assumption of Theorem 5.2, the problem (22)–(23) has a solution* $U_h^{(n)}$ *which quenches in a finite time* $T_h^{\Delta t}$ *and the following relation holds*

$$\lim_{h\to 0} T_h^{\Delta t} = T_q.$$

Proof. We know from Remark 5.1 that $\frac{\tau}{1-(1-\tau^{'})^{p+1}}$ is bounded. Letting $0 < \varepsilon < T_q/2$, there exists a constant $R \in (0,1)$ such that

$$\frac{\tau R^{p+1}}{1-(1-\tau')^{p+1}} < \frac{\varepsilon}{2}. \tag{37}$$

Since u quenches at the time T_q, there exist $T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ and $h_0(\varepsilon) > 0$ such that $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [T_1, T_q)$, $h \leq h_0(\varepsilon)$. Let q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_q)$ for $h \leq h_0(\varepsilon)$. It follows from Theorem 5.1 that the problem (22)–(23) has a solution $U_h^{(n)}$ which obeys $\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{R}{2}$ for $n \leq q$, $h \leq h_0(\varepsilon)$, which implies that

$$\|U_h^{(q)}\|_{\inf} \leq \|U_h^{(q)} - u_h(t_q)\|_\infty + \|u_h(t_q)\|_{\inf} < \frac{R}{2} + \frac{R}{2} = R, \quad h \leq h_0(\varepsilon).$$

From Theorem 5.2, $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark 5.1 and (37) that $|T_h^{\Delta t} - t_q| \leq \frac{\tau \|U_h^{(q)}\|_{\inf}^{p+1}}{1-(1-\tau')^{p+1}} < \frac{\varepsilon}{2}$ because $\|U_h^{(q)}\|_{\inf} < R$ for $h \leq h_0(\varepsilon)$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T_q - T_h^{\Delta t}| \leq |T_q - t_q| + |t_q - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

which leads us to the result. □

Remark 5.2. *Consider the problem (1), (3) for $-1 < x < 1$, $t > 0$ with Dirichlet boundary conditions*

$$u(-1, t) = 1, \quad u(1, t) = 1,$$

where $p > 0$, $u_0 \in C^1([-1, 1])$, $u_0'(-1) = u_0'(1) = 0$, $u_0(x)$ is symmetric in $[-1, 1]$, $u_0'(x) \geq 0$ in $[0, 1]$.
From the maximum principle, u is symmetric in t. To obtain an approximation of the quenching time for the classical solution u of the above problem, it suffices to get the one of the classical solution v of the problem (1), (3) with boundary conditions

$$v_x(0, t) = 0, \quad v(1, t) = 1, \quad t > 0.$$

Approximate v by the solution $V_h(t)$ of the following semidiscrete scheme

$$\frac{d}{dt} V_i(t) = \delta^2 V_i(t) - b_i V_i^{-p}(t), \quad 0 \leq i \leq I - 1,$$

$$V_I(t) = 1, \quad V_i(0) = \varphi_i > 0, \quad 0 \leq i \leq I,$$

where $\varphi_{i+1} \geq \varphi_i$, $0 \leq i \leq I - 1$. We easily prove that $V_{i+1}(t) \geq V_i(t)$, $0 \leq i \leq I-1$. Let us notice that to establish the convergence of the semidiscrete quenching time, it suffices to take $J_i(t) = \frac{dV_i(t)}{dt} + A(1 - ih)V_i^{-p}(t)$, $0 \leq i \leq I$ and one gets without difficulty an estimate as in (12). If we consider a discrete form, to establish an estimate as in (35), one may take $J_i^{(n)} = \delta_t V_i^{(n)} + A(1 - ih)(V_i^{(n)})^{-p}$, $0 \leq i \leq I$. On the other hand, one easily obtains the other results with a slight modification of the methods developed in the paper.

6 Numerical results

In this section, we present some numerical approximations to the quenching time for the solution of the problem (1)–(3) in the case where $p = 1$ and $u_0(x) = \frac{2+\varepsilon\cos(\pi x)}{4}$ with $0 < \varepsilon \leq 1$. Firstly, we take the explicit scheme in (22)–(23). Secondly, we use the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - b_i (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I,$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = K \|U_h^{(n)}\|_{\inf}^{p+1}$ with $K = 10^{-3}$.

In both cases, $\varphi_i = \frac{2+\varepsilon\cos(\pi i h)}{4}$, $0 \leq i \leq I$. For the above implicit scheme, the existence and positivity of the discrete solution $U_h^{(n)}$ is guaranteed using standard methods (see [3]). In the tables 1–8, in rows, we present the numerical quenching times, the numbers of iterations and the CPU times corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

Table 1: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method for $\varepsilon = 1$

I	t_n	n	*CPU time*
16	0.062132	4102	1
32	0.062253	15883	3
64	0.062312	61257	60
128	0.062322	235525	1245

Table 2: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 1$

I	t_n	n	*CPU time*
16	0.062302	4017	1
32	0.062317	15499	6
64	0.062323	59679	138
128	0.062324	229179	4260

Table 3: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method for $\varepsilon = 1/10$

I	t_n	n	$CPU\ time$
16	0.121368	2389	4
32	0.121210	8882	16
64	0.121170	32769	222
128	0.121157	119887	3887

Table 4: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 1/10$

I	t_n	n	$CPU\ time$
16	0.121316	14047	25
32	0.121326	14071	45
64	0.121328	14091	168
128	0.121329	14098	795

Table 5: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method for $\varepsilon = 1/100$

I	t_n	n	$CPU\ time$
16	0.124875	2356	3
32	0.124694	8728	17
64	0.124649	32091	236
128	0.124638	112964	3974

Table 6: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 1/100$

I	t_n	n	$CPU\ time$
16	0.124822	13915	24
32	0.1248195	13920	44
64	0.1248193	13923	168
128	0.1248191	13925	793

Table 7: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method for $\varepsilon = 1/1000$

I	t_n	n	$CPU\ time$
16	0.125208	2351	3
32	0.125024	8708	17
64	0.124979	32006	191
128	0.124957	112873	3852

Table 8: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method for $\varepsilon = 1/1000$

I	t_n	n	$CPU\ time$
16	0.125155	13914	26
32	0.12515090	13917	52
64	0.12515091	13918	154
128	0.12515093	13919	781

Remark 6.1. *When* $\varepsilon = 0$ *and* $p = 1$*, we know that the quenching time of the continuous solution of (1)–(3) is equal* 0.125*. We have also seen in Remark 3.3 that the quenching time of the semidiscrete solution is equal* 0.125*. We observe from Tables 1–8 that when* ε *decays to zero, then the numerical quenching time of the discrete solution goes to* 0.125*.*

In the following, we also give some plots to illustrate our analysis. For the different plots, we have used both implicit and explicit schemes in the case where $I = 1/16$, $\varepsilon = 1$.
In Figures 1 and 2, we can appreciate that the discrete solution is nonincreasing and reaches the value zero at the last node.

In Figures 3 and 4, we see that the approximation of $u_{\min}(t)$ is nonincreasing and reaches the value zero at the time $t \simeq 0.062$.

In figures 5 and 6, we observe that the approximation of $u(x, T)$ is nonincreasing and reaches the value zero at the last node. Here, T is the quenching time of the solution u.

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 12, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper.

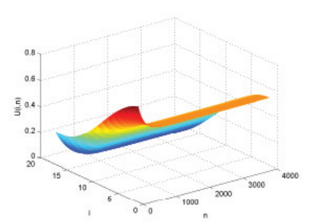

FIGURE 1 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 16$ (implicit scheme).

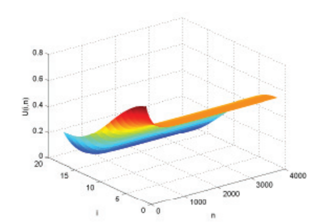

FIGURE 2 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 16$ (explicit scheme).

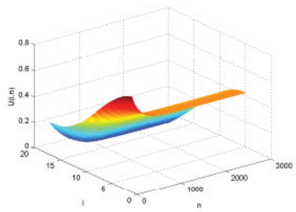

FIGURE 3 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{100}$ $I = 16$ (implicit scheme).

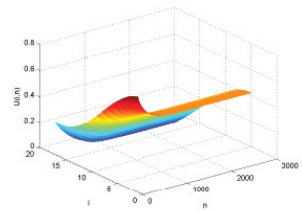

FIGURE 4 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{100}$ $I = 16$ (explicit scheme).

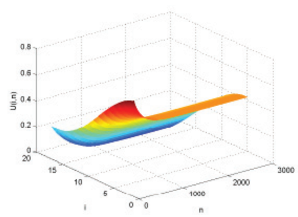

FIGURE 5 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 32$ (implicit scheme).

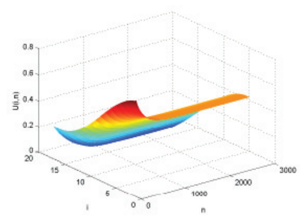

FIGURE 6 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 326$ (explicit scheme).

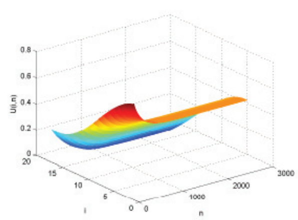

FIGURE 7 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{100}$ $I = 32$ (implicit scheme).

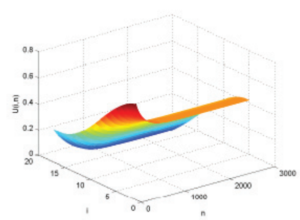

FIGURE 8 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 32$ (explicit scheme).

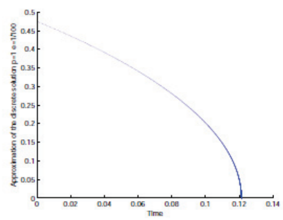

FIGURE 9 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 16$ (implicit scheme).

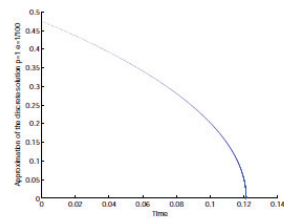

FIGURE 10 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{10}$ $I = 16$ (explicit scheme).

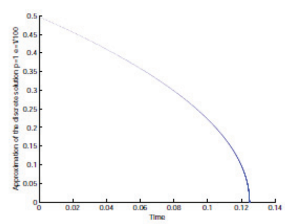

FIGURE 11 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{100}$ $I = 16$ (implicit scheme).

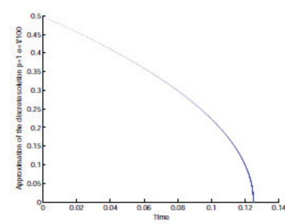

FIGURE 12 – Evolution of the discrete solution, source $\varepsilon = \frac{1}{100}$ $I = 16$ (explicit scheme).

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