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ON THE EXPONENTIAL DIOPHANTINE EQUATION $p^x - 2^y = z^2$ WITH $p = k^2 + 2$, A PRIME NUMBER

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Abstract

We are interested in finding non-negative integer solutions for the Diophantine equation $p^x - 2^y = z^2$, where $p = k^2 + 2$ is a prime number and $k \geq 0$. We show that all the positive integer solutions of this equation are given by $(1, 1, k)$ if $p \geq 11$, $(1, 1, 1), (3, 1, 5), (2, 3, 1)$ if $p = 3$. In the case $p = 2$ the equation has two infinite and disjoint families of solutions. The proofs are based on the use of the Catalan-Mihăilescu Theorem (old Catalan conjecture) and properties of the modular arithmetic. In addition, we prove that equations of type $p^x - 2^y = w^{2^u}$ with $u \geq 2$ do not have positive integer solutions if $p \geq 11$ and k is not a perfect square. Moreover, we find exactly two positive integer solutions for $p^x - 2^y = w^{2^u}$, with $u \geq 2$, when $p = 3$.

1 Introduction

Diophantine equations of the form $a^x + b^y = c^z$ have been studied by numerous mathematicians for many decades and by a variety of methods. One of the

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first references to these equations was given by Fermat-Euler [2], showing that $(a, c) = (5, 3)$ is the unique positive integer solution of the equation $a^2+2 = c^3$. Scott [7] proved that if $a > 1$ and $b > 1$ satisfy $\gcd(a, b) = 1$ and c is prime, then the equation $a^x + b^y = c^z$ has at most two solutions in positive integers (x, y, z), when $c \neq 2$, and at most one solution (x, y, z) when $c = 2$, except for two cases (taking $a < b$): $(a, b, c) = (3, 5, 2)$, which has exactly three solutions $(x, y, z) = (1, 1, 3), (3, 1, 5), (1, 3, 7)$ and $(a, b, c) = (3, 13, 2)$, which has exactly two solutions $(x, y, z) = (1, 1, 4), (5, 1, 8)$ ([3], Section D9, p. 87). In 2007, Acu [1] solved the equation for $a = 2$, $b = 5$ and $z = 2$. The non-negative integer solutions to the equation are $(x, y, c) \in \{(3, 0, 3), (2, 1, 3)\}$. In 2011, Suvarnamani [10] studied the Diophantine equation $2^x + p^y = z^2$. Rabago [6] studied the equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. He found exactly two solutions (x, y, z) in non-negative integers for each one. The solution sets are $\{(1, 0, 2),(4, 1, 10)\}$ and $\{(1, 0, 2),(2, 1, 10)\}$, respectively. A. Suvarnamani *et al.* [9] found solutions of two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. In 2019, Thongnak *et al.* found exactly two non-trivial solutions for the equation $2^x - 3^y = z^2$, namely $(1, 0, 1)$ and $(2, 1, 1)$. In this paper, we use elementary methods and Catalan-Mihăilescu Theorem (Theorem 2.1) to study exponential Diophantine equations of the form $p^x - 2^y = z^2$, where $p = k^2 + 2$ are prime numbers, $(x, y, z) \in \mathbb{N}^3$ e $k \in \mathbb{N}$.

2 Notation and Preliminary Results

Denote by $\mathbb{Z}$ be the set of integer numbers and let $\mathbb{N}$ be the set of all positive integers together with the number 0, that is, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, such a set will be called the set of *natural numbers*. Define $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\mathbb{N}^q = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ as the *cartesian product* of q copies of $\mathbb{N}$. We will use the $\equiv$ symbol for congruence module m and $a \equiv b \pmod{m}$ means that a is congruent to b module m. The set of all non-negative integer solutions of the equation $p^x - 2^y = z^2$ will be said simply the *solution set of the equation*, i. e., the set $\{(x, y, z) \in \mathbb{N}^3 \,|\, p^x - 2^y = z^2\}$.

The following theorem was proved by Mihăilescu in [4] and is written in the form of his famous conjecture.

Theorem 2.1. *(Catalan-Mihăilescu Theorem)* $(3, 2, 2, 3)$ *is the unique solution* $(a, b, x, y) \in \mathbb{N}^4$ *for the Diophantine equation* $a^x - b^y = 1$ *where each* $a, b, x, y > 1$.

Remark 2.2. *The equation* $a^x - z^2 = 1$ *has no positive integer solutions if* $a, x, z > 1$.

We need the important result obtained by Sury [8]. This result was obtained first by Nagell [5], but the proof is no elementary, while the Sury's proof is elementary.

Theorem 2.3. *The Diophantine equation* $z^2 + 2 = y^x, x > 1$ *has only the solutions* $(z, y, x) = (\pm 5, 3, 3)$.

The next lemma has easy proof. It will be used in the proofs of the main theorems.

Lemma 2.4. *There is no* $w \in \mathbb{Z}$ *such that* $w^2 \equiv 3 \pmod 4$.

3 Main Theorems

The following results were divided into three main parts, one for prime numbers ≥ 11 and the other two for prime numbers 3 and 2.

Theorem 3.1. *If* $p = k^2 + 2$ *is a prime number for some* $k \geq 3$, *then the solution set of the Diophantine equation*

$$p^x - 2^y = z^2 \tag{1}$$

is given by $\{(0, 0, 0), (1, 1, k)\}$.

Proof. The proof will be done by testing several cases. We will assume that x, y, z are natural numbers such that they satisfy the equation $p^x - 2^y = z^2$.

Case 1. $(x = 0)$. We will divide this case into the subcases $y = 0$ and $y \geq 1$.

Case i: If $y = 0$, then $z^2 = 0$ and so $(0, 0, 0)$ is a solution of the equation.

Case ii: If $y \geq 1$, then $z^2 = 1 - 2^y \leq -1$ which is an absurd.

Case 2. $(x = 1)$. In this case we have $p - 2^y = z^2$. Let us divide this case into the subcases $y = 0$, $y = 1$ and $y \geq 2$.

Case i: If $y = 0$, then $p - 1 = z^2 = k^2 + 1$, so we have $(z - k) \cdot (z + k) = 1$, whence we conclude that $k = 0$, which is a contradiction with $k \geq 3$.

Case ii: If $y = 1$, then $p - 2 = z^2 = k^2$, so $z = k$ and a solution to the equation is equal to $(1, 1, k)$.

Case iii: In the last subcase one rewrite $p - 2^y = z^2$ as $z^2 - k^2 = 2 - 2^y$ from which one obtain $(z - k) \cdot (z + k) = 2 \cdot (1 - 2^{y-1})$. Since z and k are both odd numbers, there are non-zero integers a, b such that $z - k = 2a$ and $z + k = 2b$. It follows that $2ab = 1 - 2^{y-1}$ which is an absurd since the left side is even and the right side is odd because $y \geq 2$.

Therefore, $(1, 1, k)$ is the only solution to the equation (1) in this case.

Case 3. ($x > 1$, $y = 0$). In this case, the equation (1) is reduced to $p^x - z^2 = 1$. Now one apply Remark 2.2 to conclude that $z \in \{0, 1\}$. If $z = 0$, we have $p^x = 1$ which implies that $x = 0$, a contradiction. In the latter subcase one obtain $p^x = 2$ which is an absurd since $p \geq 11$. Therefore there is no solution of (1) in this case.

Case 4. ($x > 1$ and $y = 1$). In this case the equation (1) is reduced to $p^x - 2 = z^2$ which is a contradiction with Theorem 2.3. Therefore, there is no solution of (1) in this case.

Case 5. ($x > 1$ even and $y > 1$). We will show that this case also has no positive integer solutions. In this case there exists $t \in \mathbb{N}^*$ such that

$$p^{2t} - z^2 = 2^y \Longrightarrow \big(p^t - z\big) \cdot \big(p^t + z\big) = 2^y.$$

Since $p^t - z < p^t + z$ are both even numbers, there exists $\alpha \in \mathbb{N}$ such that $p^t - z = 2^\alpha$ and $p^t + z = 2^{y-\alpha}$ from which one obtain

$$2p^t = 2^\alpha + 2^{y-\alpha} = 2^\alpha(1 + 2^{y-2\alpha}). \tag{2}$$

Let us divide this case into the subcases $\alpha = 0, \alpha = 1$ and $\alpha \geq 2$.

If $\alpha = 0$, we have $2p^t = 1 + 2^y$ which is an absurd for $y > 1$. If $\alpha \geq 2$, we have $p^t = 2^{\alpha-1}(1 + 2^{y-2\alpha})$ which is also an absurd because the left side is an odd number and the right one is even. If $\alpha = 1$, the equality (2) is reduced to $p^t - 2^{y-2} = 1$. Now one apply Theorem 2.1 to conclude that $t = 1$ or $y \in \{2, 3\}$. We divide the subcase $\alpha = 1$ into the two subcases.

Case i: If $t = 1$, then $2^{y-2} = p - 1 = k^2 + 1$. If $y \in \{2, 3\}$, we have $k^2 + 1 \in \{1, 2\}$ which implies that $k \in \{0, 1\}$, which is a contradiction with $k \geq 3$. Now, we suppose $y \geq 4$. In this subcase we have $2^{y-2} \equiv 0 \pmod 4$ and therefore $k^2 \equiv 3 \pmod 4$, which is a contradiction with Lemma 2.4.

Case ii: If $y \in \{2, 3\}$, then $p^t \in \{2, 3\}$, which is a contradiction because $p^t \geq 11$.

Case 6. ($x > 1$ odd and $y > 1$). In this case the equality (1) can be rewritten as $p^{2s+1} - 2^y = z^2$, for some $s \geq 1$. Since p is an odd prime it follows that either $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$, moreover $2^y \equiv 0 \pmod 4$ because $y \geq 2$. If $p \equiv 1 \pmod 4$, then $k^2 \equiv 3 \pmod 4$ and by Lemma 2.4 we have a contradiction. Suppose $p \equiv 3 \pmod 4$. In this case

$$p^{2s} \equiv 9^s \equiv 1 \pmod 4 \Rightarrow p^{2s+1} \equiv 3 \pmod 4 \Rightarrow z^2 \equiv 3 \pmod 4,$$

an absurd by Lemma 2.4. Therefore, there is no solution of (1) in this case. $\square$

Corollary 3.2. *Let $p = k^2 + 2$ be a prime number for some integer $k \geq 3$. If k is not a perfect square, then $(0,0,0)$ is the unique solution of the Diophantine equation*

$$p^x - 2^y = w^{2^u}, (x,y,w) \in \mathbb{N}^3 \text{ and } u \geq 2. \tag{3}$$

Proof. We write $p^x - 2^y = (w^{2^{u-1}})^2, u \geq 2$. According to Theorem 3.1 we have $w^{2^{u-1}} = 0$ or $w^{2^{u-1}} = k$. In the first case, $w = 0$, thus finding the trivial solution for equation (3). In the second case, we conclude that k is a perfect square, a contradiction. Therefore, $p^x - 2^y = w^{2^u}$ has no non-trivial solution. □

Corollary 3.3. *Let $p = k^2 + 2$ be a prime number for some integer $k \geq 3$. If k is a perfect square, then $\{(0,0,0),(1,1,\sqrt{k})\}$ is the solution set of the Diophantine equation $p^x - 2^y = w^4$.*

Proof. As in Corollary 3.2, $w^2 \in \{0,k\}$ so $w = 0$ and $w = \sqrt{k}$, and the result follows. □

As an example if $p = 83$, we have $k = 9$ and the only non-negative integer solutions of $83^x - 2^y = w^4$ are $(0,0,0)$ and $(1,1,3)$. If $p = 11$, we have $k = 3$ and $(0,0,0)$ is the unique non-negative integer solution of $11^x - 2^y = w^4$.

Remark 3.4. The proofs of the results below are similar to the proofs of the Theorem 3.1 and its corollaries, we will detail only the situations that are not similar. The equation $2^x - 3^y = z^2$ was studied in [11].

Theorem 3.5. *The set $\{(0,0,0);(1,1,1);(3,1,5);(2,3,1)\}$ is the solution set of the Diophantine equation*

$$3^x - 2^y = z^2, (x,y,z) \in \mathbb{N}^3. \tag{4}$$

Proof. Let $(x,y,z) \in \mathbb{N}^3$ be a solution of the equation (4).

Similarly to the Cases 1 and 2 of Theorem 3.1 one obtain that $(0,0,0)$ is the unique solution of (4) in the case $x = 0$, $(1,1,1)$ is the unique solution of (4) in the case $x = 1$ and there is no solution of the equation (4) in the Case 3 $(x > 1, y = 0)$ and in the Case 6 ($x > 1$ odd and $y > 1$).

Case 4 ($x > 1$ and $y = 1$). In this case the equation (4) is reduced to $3^x - 2 = z^2$. For [8] the only positive integer solution to this equation is $(x,z) = (3,5)$. So we have a third solution of the equation (4) given by $(3,1,5)$.

It remains for us to analyze the analogue of Case 5, for that we will use the same notations of what were done before.

Case 5 ($x > 1$ even and $y > 1$). Take $x = 2t, t > 0$. In this case the equation is reduced to $3^{2t} - 2^y = z^2$. As before, we should have $\alpha \neq 0$ and $\alpha \ngeq 2$, so

$\alpha = 1$. So we find the following equation $3^t - 2^{y-2} = 1$. This equation can only be solved with positive integers if $t = 1$ or $y \in \{2, 3\}$. If $y = 2$, we have $3^t = 2$ a contradiction. If $y = 3$, we have $3^t = 3$ and so $t = 1$. Therefore, $x = 2$ and so we have $z^2 = 1$ which implies that $z = 1$. We found the last solution which is $(2, 3, 1)$. $\square$

Corollary 3.6. *The solution set of the Diophantine equation* $3^x - 2^y = w^{2^u}$, *with* $u \geq 2$, *is* $\{(0, 0, 0); (1, 1, 1); (2, 3, 1)\}$.

Theorem 3.7. *The solution set of the Diophantine equation*

$$2^x - 2^y = z^2, (x, y, z) \in \mathbb{N}^3, \tag{5}$$

is the disjoint union $\mathcal{A} \dot{\cup} \mathcal{B}$ *where* $\mathcal{A} = \{(s, s, 0) | s \in \mathbb{N}\}$ *and* $\mathcal{B} = \{(2s + 1, 2s, 2^s) | s \in \mathbb{N}\}$.

Proof. If $y = 0$, we have the equation $2^x - z^2 = 1$ which by Theorem 2.1 has no solution if $x > 1$ and $z > 1$. So we will have a solution only if $x \in \{0, 1\}$ or $z \in \{0, 1\}$. If $x = 0$, we have $z^2 = 0$ which implies that $z = 0$, therefore $(0, 0, 0)$ is a solution of the equation. If $x = 1$, we get $z^2 = 1$ from which we conclude that $z = 1$, so $(1, 0, 1)$ is the second solution. The cases where $z \in \{0, 1\}$ give the same solutions as found so far.

Let $y = 1$. In this case the equation (5) is reduced to $2^x - 2 = z^2$. For Sury ([8]) there are no non-negative integer solutions if $x \geq 2$. Thus $x \in \{0, 1\}$. If $x = 0$, we have $z^2 = -1$, a contradiction. If $x = 1$, we have $z^2 = 0$ therefore $(1, 1, 0)$ is the third solution.

Now consider $y > 1$. Let's divide it into two subcases.

Case i) If $x = y$. In this case $z^2 = 0$ which implies that $z = 0$, so $(x, x, 0)$ they are solutions of the equation with $x > 1$.

Case ii) If $x > y > 1$ (the case $x < y$ does not occur because in this case $z^2 < 0$, a contradiction). Since $z^2 = 2^x - 2^y$, we have $z > 0$. In this case the equation is reduced to $2^y \cdot (2^{x-y} - 1) = z^2$. Since $2^{x-y} - 1$ is odd we have $y = 2s, s \geq 1$, because z is a positive integer. We can rewrite the equation as follows $2^{2s} \cdot (2^{x-2s} - 1) = z^2$, therefore $z = a \cdot 2^s$ where $a \geq 1$ and $a^2 = 2^{x-2s} - 1$. Rewriting the previous equality as $a^2 - 2^{x-2s} = -1$, we observe that it will have no solutions if $(x - 2s) > 1$ and $a > 1$ (again applying Theorem 2.1). So it remains to analyze the cases $a = 1$ and $x - 2s = 1$. If $a = 1$, we have $z = 2^s$ and hence

$$2^{2s} \cdot (2^{x-2s} - 1) = 2^{2s} \Rightarrow 2^{x-2s} = 2 \Rightarrow x = 2s + 1.$$

The other case that we should analyze gives the same solutions. So another family of solutions is given by the triples $(2s+1, 2s, 2^s)$, $s \geq 0$. We have proven that the solution set of the equation (5) is contained in the set $\mathcal{A}\dot{\cup}\mathcal{B}$. Finally, every element of $\mathcal{A}\dot{\cup}\mathcal{B}$ is a solution of the equation (5). □

Corollary 3.8. *The solution set of the Diophantine equation* $2^x - 2^y = w^4$*,* $(x, y, w) \in \mathbb{N}^3$*, is the disjoint union* $\mathcal{A}\dot{\cup}\mathcal{C}$ *where* $\mathcal{A} = \{(t, t, 0)|t \in \mathbb{N}\}$ *and* $\mathcal{C} = \{(4t + 1, 4t, 2^t)|t \in \mathbb{N}\}$.

Proof. Clearly every element of $\mathcal{A}\dot{\cup}\mathcal{C}$ is a solution of the equation $2^x - 2^y = w^4$. Reciprocally, let $(\widehat{x}, \widehat{y}, \widehat{w}) \in \mathbb{N}^3$ be a solution of $2^x - 2^y = w^4$. If one write $\widehat{z} = \widehat{w}^2$ then $(\widehat{x}, \widehat{y}, \widehat{z})$ is a solution of the equation $2^x - 2^y = z^2$, $(x, y, z) \in \mathbb{N}^3$. It follows from Theorem 3.7 that either $\widehat{z} = 0$ and $\widehat{x} = \widehat{y}$ or $\widehat{z} > 0$ and there exists $t \in \mathbb{N}$ such that $\widehat{z} = 2^{2t}$, $\widehat{y} = 4t$, $\widehat{x} = 4t + 1$. In any case, $(\widehat{x}, \widehat{y}, \widehat{w}) \in \mathcal{A}\dot{\cup}\mathcal{C}$. □

References

[1] D. Acu. *On a diophantine equation*, General Mathematics, **(15)** (2007) 145-148.

[2] L. Euler, “Elements of Algebra”, Vol. 2, 1822.

[3] R. Guy, “Unsolved Problems in Number Theory”, Springer-Verlag, 1981.

[4] P. Mihăilescu, *Primary ciclotomic units and a proof of Catalan's conjecture*, Journal fúr die reine and angewandte Mathematik, **(27)** (2004) 167–195.

[5] T. Nagell, *Verallgemeinerung eines Fermatschen Satzes*, Arch. Math. **5**, (1954) 153–159.

[6] J. F. T. Rabago, *On two Diophantine equation* $3^x + 19^y = z^2$ *and* $3^x + 91^y = z^2$, International Journal of Mathematics and Scientific Computing, **(3)** n.1 (2013) 28-29.

[7] R. Scott, *On the Equations* $p^x - b^y = c$ *and* $a^x + b^y = c^z$, Journal of Number Theory, **44**(2) (1993) 153–165.

[8] B. Sury, *On the Diophantine equation* $x^2 + 2 = y^n$, Archiv der Mathematik, **(74)** (2000) 350–355.

[9] A. Suvarnamani, A. Singta and S. Chotchaisthit, *On two Diophantine equations* $4^x + 7^y = z^2$ *and* $4^x + 11^y = z^2$, Science and Technology RMUTT Journal, **(1)** (2011) 25-28.

[10] A. Suvarnamani, *Solutions of the Diophantine equations* $2^x + p^y = z^2$, International Journal of Mathematical Sciences and Applications, **(1)**, n.3 (2011), 1415-1419.

[11] S. Thongnak, W. Chuayjana, T. Kaewong. *On the exponential diophantine equation* $2^x - 3^y = z^2$, Southeast-Asian J. of Sciences: Vol. 7, n. 1 (2019) 1–4.