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ON THE RATE OF CONVERGENCE IN LIMIT THEOREMS FOR GEOMETRIC SUMS

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Abstract

Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables. Let $N_q, q \in (0, 1)$ be a geometric random variable with probabilities $P(N_q = k) = q(1-q)^{k-1}, k = 1, 2, \ldots$. Moreover, suppose that $N_q, q \in (0, 1)$ is independent of all $X_{nj}, j = 1, 2, \ldots; n = 1, 2, \ldots$. Let $S_{N_q} = X_{n1} + X_{n2} + \ldots + X_{nN_q}$ denote the geometric sum of independent identically distributed random variables $X_{nj}, j = 1, 2, \ldots; n = 1, 2, \ldots$ (by convention, $S_0 = 0$). The main purpose of this article is to establish the rate of convergence in some Renyi-type limit theorems for geometric sums via Trotter-operator method.

1 Introduction

Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with finite mean $E(X_{nj}) = m$ and $0 < D(X_{nj}) = \sigma^2 < +\infty, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Let $N_q \sim Geo(q), q \in (0, 1)$ be a geometric random variable with probabilities $P(N_q = k) = q(1-q)^{k-1}, k = 1, 2, \ldots$. Moreover, suppose that the N_q is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. It it to be noticed that N_q and all random variables $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$ are defined on the same probability

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space $(\Omega, \mathbb{A}, \mathbb{P})$. We will denote by S_{N_q} the geometric sum

$$S_{N_q} = X_{n1} + X_{n2} + \ldots + X_{nN_q}, \quad S_0 = 0 \quad \text{by convention.} \tag{1}$$

Up to the present one of well-known limit theorems for geometric sums is the Renyi's limit theorem (see Theorem 8.1.5 page 246, [14]). By its statement, let $(X_j, j = 1, 2, \ldots)$ be a sequence of independent and identically distributed random variables with common positive mean $E(X_j) = m < +\infty, j = 1, 2, \ldots$. Then,

$$q(X_1 + X_2 + \ldots + X_{N_q}) \xrightarrow{d} Z^{(m)}, \quad \text{as} \quad q \to 0^+, \tag{2}$$

where $Z^{(m)}$ is an exponential distributed random variable with positive mean $E(Z^{(m)}) = m$ and symbol $\xrightarrow{d}$ is denoted by the convergence in distribution.

During the last several decades the limit theorems for geometric sums have risen to become one of the most important problems having the deep applications to insurance risk theory, stochastic finance, queueing theory (see [2], [3], [9], [14], [13], [8]). Moreover, it is well known that characteristic function have been used in study of Renyi's limit theorem as a power mathematical tool (see for instance [16], [14]).

The goal of this paper is to establish the rates of convergence in some Renyi-type limit theorems for geometric sums of row-wise triangular array independent identically distributed random variables via Trotter-operator method. It is worth pointing out that all proofs of theorems of this paper utilize Trotter's idea from Trotter [20] and the method used in this paper is the same as in works of Renyi [16], Butzer, Hahn, and Westphal [4], [5] and [6], Rychlick and Szynal [17] and [18], Cioczek and Szynal [7], Hung and Thanh [12]. Note that the effect of Trotter-operator method also is powerful in cases of the limit theorems for random sums of multidimensional random variables (in the multidimensional case, the reader may refer to [15], [19], [5], [10]).

The rest of the paper is organized as follows. Some notations, definitions and properties needed in this paper will be presented in Section 2. The Section 3 is devoted to the rates of convergence in some Renyi-type limit theorems for geometric sums via Trotter-operator method.

2 Preliminaries

Throughout this paper, the symbols $C_B(\mathbb{R})$ will denote the set of all bounded uniformly continuous functions on $\mathbb{R}$ and

$$C_B^r(\mathbb{R}) := \{f \in C_B(\mathbb{R}) : f^{(j)} \in C_B(\mathbb{R}), j = 1, 2, \ldots, r\}, r \in \mathbb{N}.$$

Note that $C_B(\mathbb{R}) = C_B^o(\mathbb{R})$, by convention. The norm of function $f \in C_B(\mathbb{R})$ is defined by $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$.

For the purpose of the present paper, we will recall definition of Trotter's operator (see [20], for the definition of Trotter operator).

Definition 2.1. *(Trotter [20], 1959) Let X be a random variable. A linear operator $T_X : C_B(\mathbb{R}) \to C_B(\mathbb{R})$, is said to be Trotter operator and it is defined by*

$$T_X f(t) := Ef(X+t) = \int_{\mathbb{R}} f(x+t)dF_X(x), \quad t \in \mathbb{R}, f \in C_B(\mathbb{R}), \tag{3}$$

where F_X is the distribution function of X.

In the sequel, we shall use the following properties of Trotter operator T_X in (3) (we refer the reader to [20], [16], [4], for more details).

1. The operator T_X is a linear positive "contraction" operator, i.e.,

$$\| T_X f \| \leq \| f \|,$$

for each $f \in C_B(\mathbb{R})$.

2. The operators T_{X_1} and T_{X_2} commute.

3. The equation $T_X f(t) = T_Y f(t)$ for $f \in C_B(\mathbb{R}), t \in \mathbb{R}$, provided that X and Y are identically distributed random variables.

4. If $X_1, X_2, \ldots, X_n$ are independent random variables, then for $f \in C_B(\mathbb{R})$

$$T_{X_1+\ldots+X_n}(f) = T_{X_1} \ldots T_{X_n}(f).$$

5. Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables (in each group) and they are independent. Then for each $f \in C_B(\mathbb{R})$

$$\|T_{X_1+\ldots+X_n}(f) - T_{Y_1+\ldots+Y_n}(f)\| \leq \sum_{i=1}^{n} \|T_{X_i}(f) - T_{Y_i}(f)\|.$$

Furthermore, for two independent random variables X and Y, for each $f \in C_B(\mathbb{R})$ and $n = 1, 2, \ldots$

$$\| T_X^n(f) - T_Y^n(f) \| \leq n \| T_X(f) - T_Y(t) \| .$$

6. Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, are independent random variables (in each group) and they are independent. Moreover, assume that $N_1, N_2, \ldots, N_n, \ldots$ are positive integer-valued random variables independent of all X_j and $Y_j, j = 1, 2, \ldots$. Then, for each $f \in C_B(\mathbb{R})$

$$\|T_{X_1+\ldots+X_{N_n}}(f) - T_{Y_1+\ldots+Y_{N_n}}(f)\| \leq \sum_{n=1}^{\infty} P(N_n = n) \sum_{i=1}^{n} \|T_{X_i}(f) - T_{Y_i}(f)\|.$$

7. If
$$\lim_{n\to\infty} \parallel T_{X_n}(f) - T_X(f) \parallel = 0, \forall f \in C_B^r(\mathbb{R}), r \in \mathbb{N},$$
then $X_n \xrightarrow{d} X$ as $n \to \infty$.

It is to be noticed that during the last several decades the operator method has risen to become one of the important most tools available for studying with certain types of large scale problems as limit theorems for independent random variables. And Trotter (1959, [20]) was one of mathematicians who succeeded in using the operator-method in order to get elementary proofs in central limit theorem for sums of independent random variables. The Trotter's idea have been used in many areas of probability theory and related fields. For a deeper discussion of Trotter- operator method we refer the reader to [20], [16], [4], [5], [6], [17], [18], [7].

Before stating the main results we first need to recall the definition of the modulus of continuity and Lipschitz classes.

Definition 2.2. *([4], [5]) For any $f \in C_B(\mathbb{R})$, the modulus of continuity with $\delta > 0$, is defined by*

$$\omega(f, \delta) = \sup_{|h|\leq\delta} \{|f(t+h) - f(t)|, t \in \mathbb{R}\}. \tag{4}$$

We shall need in the sequel some properties of the modulus of continuity $\omega(f, \delta)$ from (4).

1. The modulus of continuity $\omega(f, \delta)$ is a monotone decreasing function of δ with $\omega(f, \delta) \to 0$ for $\delta \to 0^+$.

2. For $t \geq 0$, we have $\omega(f, t\delta) \leq (1 + t)\omega(f, \delta)$.

The detailed proofs of the properties of the modulus of continuity can be found in [4] and [5].

Definition 2.3. *([4], [5]) A function $f \in C_B(\mathbb{R})$, is said to satisfy a Lipschitz condition of order $\alpha, 0 < \alpha \leq 1$, in symbols $f \in Lip(\alpha)$ if*

$$\omega(f; \delta) = O(\delta^\alpha). \tag{5}$$

It is obvious that $f^{'} \in C_B(\mathbb{R})$ implies $f \in Lip(1)$.

The proofs of some limit theorems in this article base upon the Taylor series expansion for every function $f \in C_B^r(\mathbb{R})$

$$f(x+y) = \sum_{j=0}^{r} \frac{x^j f^{(j)}(y)}{j!} + \frac{x^r}{(r-1)!}\left(f^{(r)}(\eta) - f^{(r)}(y)\right), \tag{6}$$

where $\mid \eta - y \mid \leq x$.

3 Main Results

In the remaining part of this paper, we will investigate some limit theorems for geometric sums of a row-wise triangular array independent identically distributed random variables. The received results in this part confirmed that the convergence rate in Renyi-type limit theorems for geometric sums (see [13] for more details) could be established by Trotter-operator method, too. The following lemma states one of the most important properties of random geometric sums.

Lemma 3.1. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent, exponential distributed random variables with positive parameter p. Moreover, let $N_q, q \in (0, 1)$ be a geometric random variable with parameter $q, q \in (0, 1)$, and independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Then, for $n = 1, 2, \ldots$, the geometric sum $S_{N_q} = X_{n1} + X_{n2} + \ldots + X_{nN_q}$ is a exponential random variable with parameter pq.*

Proof. By an argument analogous to the Theorem 2.3 ([11], page 121), let us denote by $g(t) = \frac{qt}{1-(1-q)t}$ the generating function of N_q and by $\varphi(t) = \frac{p}{p-it}$ the characteristic function of X_{nj}, respectively. Then, the characteristic function of random sum S_{N_q} is defined by

$$\Psi(t) = g(\varphi(t)) = \frac{pq}{pq - it}.$$

Thus, $S_{N_q} \sim Exp(pq)$. The proof is straight-forward. □

The following theorems will strongly confirm the simplicity of Trotter-operator method in studies of Renyi-type limit theorem for row-wise triangular array of independent identically distributed random variables.

Theorem 3.1. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative independent identically distributed random variables with mean $0 < E(X_{nj}) = m, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Let N_q be a geometric random variable with parameter $q, q \in (0, 1)$. Then,*

$$qS_{N_q} \xrightarrow{d} Z^{(m)}, \quad as \quad q \to 0^+, \tag{7}$$

where $S_{N_q} = X_{n1} + X_{n2} + \ldots + X_{nN_q}$, for $n = 1, 2, \ldots$ and $Z^{(m)}$ is a exponential distributed random variable with positive mean $E(Z^{(m)}) = m$. Note that $S_0 = 0$ by convention.

Proof. Let $Z_1^{(m)}, Z_2^{(m)}, \ldots$ be a sequence of independent exponential distributed random variables with common mean m, i.e. $Z_j^{(m)} \sim Exp(\frac{1}{m})$. Moreover, let $N_q, q \in (0, 1)$ be a geometric distributed random variable with parameter q.

According to Lemma 3.1 we conclude that the geometric sum $\sum_{j=1}^{N_q} Z_j^{(m)}$ is belong to exponential law with parameter $\frac{1}{mq}$, i.e. $\sum_{j=1}^{N_q} Z_j^{(m)} \sim Exp(\frac{1}{mq})$.

Thus, our proof starts with the observation that, for an exponential distributed random variable with positive parameter $\frac{1}{m}$, denoted by Z^m, we have

$$Z^{(m)} \stackrel{d}{=} q \sum_{j=1}^{N_q} Z_j^{(m)}. \tag{8}$$

Note that the random variable $Z^{(m)}$ is defined in (8) can also be called a geometric infinitely divisible (GID) random variable (see [1], for definition of GID). Based on properties of Trotter's operator, the Theorem 3.1 will be proved if

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| = o(1), \quad \text{as} \quad q \to 0^+, \tag{9}$$

for $f \in C_B^1(\mathbb{R})$. Using the inequalities concerning with Trotter's operator, for $f \in C_B^1(\mathbb{R})$,

$$\begin{aligned} \| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| &\leq \sum_{n=1}^{\infty} P(N_n = n) n \| T_{qX_{n1}} f - T_{qZ_1^{(m)}} f \| \\ &\leq E(N_q) \| T_{qX_{n1}} f - T_{qZ_1^{(m)}} f \| = \frac{1}{q} \| T_{qX_{n1}} f - T_{qZ_1^{(m)}} f \| . \end{aligned} \tag{10}$$

Since $f \in C_B^1(\mathbb{R})$, one has by the Taylor series expansion

$$f(x+y) = f(y) + x f^{'}(y) + x[f^{'}(\eta) - f^{'}(y)], \tag{11}$$

where $| \eta - y | \leq x$.

Then, applying the Trotter operator to function $f \in C_B^1(\mathbb{R})$ in (11), this yields

$$\begin{aligned} T_{qX_{n1}} f(y) = E(f(qX_{n1} + y)) &= \int_0^{+\infty} f(qx + y) dF_{X_{n1}}(x) \\ &= f(y) + q f^{'}(y) \int_0^{+\infty} x dF_{X_{n1}}(x) + q \int_0^{+\infty} x[f^{'}(\eta_1) - f^{'}(y)] dF_{X_{n1}}(x) \\ &= f(y) + qm f^{'}(y) + q \int_0^{+\infty} x[f^{'}(\eta_1) - f^{'}(y)] dF_{X_{n1}}(x), \end{aligned} \tag{12}$$

where $| \eta_1 - y | < q | x |$.

In the same way we get

$$\begin{aligned} T_{qZ_1^{(m)}} f(y) &= E\left(f(qZ_1^{(m)} + y) \right) \\ &= f(y) + qm f^{'}(y) + q \int_0^{+\infty} x[f^{'}(\eta_2) - f^{'}(y)] dF_{Z_1^{(m)}}(x), \end{aligned} \tag{13}$$

where $\mid \eta_2 - y \mid < q \mid x \mid$.

Then, applying (12) with (13) we have

$$\begin{aligned}
&\mid T_{qX_{n1}} f(y) - T_{qZ_1^{(m)}} f(y) \mid \\
&\leq q \int_0^{+\infty} x \mid f^{'}(\eta_1) - f^{'}(y) \mid dF_{X_{n1}(x)} + q \int_0^{+\infty} x \mid f^{'}(\eta_2) - f^{'}(y) \mid dF_{Z_1^{(m)}}(x) \\
&\leq q \int_0^{\frac{\delta}{q}} x \mid f^{'}(\eta_1) - f^{'}(y) \mid dF_{X_{n1}}(x) + q \int_{\frac{\delta}{q}}^{+\infty} x \mid f^{'}(\eta_1) - f^{'}(y) \mid dF_{X_{n1}}(x) \\
&+ q \int_0^{\frac{\delta}{q}} x \mid f^{'}(\eta_2) - f^{'}(y) \mid dF_{X_{n1}}(x) + q \int_{\frac{\delta}{q}}^{+\infty} x \mid f^{'}(\eta_2) - f^{'}(y) \mid dF_{X_{n1}}(x) \\
&\leq q\epsilon \int_0^{\frac{\delta}{q}} x dF_{X_{n1}}(x) + q \int_{\frac{\delta}{q}}^{+\infty} x \mid f^{'}(\eta_1) - f^{'}(y) \mid dF_{X_{n1}}(x) + \\
&+ q\epsilon \int_0^{\frac{\delta}{q}} x dF_{Z_1}^{(m)}(x) + q \int_{\frac{\delta}{q}}^{+\infty} x \mid f^{'}(\eta_2) - f^{'}(y) \mid dF_{Z_1^{(m)}}(x) \\
&\leq mq\epsilon + 2q \parallel f^{'} \parallel \int_{\frac{\delta}{q}}^{+\infty} x dF_{X_{n1}}(x) + mq\epsilon + 2q \parallel f^{'} \parallel \int_{\frac{\delta}{q}}^{+\infty} x dF_{Z_1^{(m)}}(x).
\end{aligned} \tag{14}$$

Note that in order to estimate the integrals above we used fact that since $f \in C_B^1(\mathbb{R})$, to each $\epsilon > 0$ there exists $\delta > 0$, such that if $\mid \xi - y \mid < \frac{q}{x} < \delta$, implies $\mid f^{'}(\xi) - f^{'}(y) \mid < \epsilon$ and $\mid f^{'}(\xi) - f^{'}(y) \mid < 2 \parallel f^{'} \parallel$, for all $\mid \xi - y \mid > \frac{q}{x}$. We then infer that, for $f \in C_B^1(\mathbb{R})$,

$$\parallel T_{qX_{n1}} f - T_{qZ_1^{(m)}} f \parallel \leq 2qm\epsilon + 2q \parallel f^{'} \parallel \left(\int_{\frac{\delta}{q}}^{+\infty} x dF_{X_{n1}}(x) + \int_{\frac{\delta}{q}}^{+\infty} x dF_{Z_1^{(m)}}(x) \right), \tag{15}$$

and it follows that , for $f \in C_B^1(\mathbb{R})$,

$$\begin{aligned}
\parallel T_{qS_{N_q}} f - T_{Z^{(m)}} \parallel &\leq \frac{1}{q} \parallel T_{qX_{n1}} f - T_{qZ_1^{(m)}} \parallel \\
&\leq 2m\epsilon + 2 \parallel f^{'} \parallel \left(\int_{\frac{\delta}{q}}^{+\infty} x dF_{X_{n1}}(x) + \int_{\frac{\delta}{q}}^{+\infty} x dF_{Z_1^{(m)}}(x) \right).
\end{aligned} \tag{16}$$

Hence, on account of finiteness of $E(X_{nj}), j = 1, 2, \ldots, n; n = 1, 2, \ldots$ and $E(Z_1^{(m)})$, we obtain

$$\parallel T_{qS_{N_q}} f - T_{Z^{(m)}} f \parallel = o(1) \quad \text{as} \quad q \to 0^+.$$

The proof is complete. $\square$

Theorem 3.2. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative independent identically distributed random variables with mean $E(\mid X_{n1} \mid^k) < +\infty, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots$. Let N_q be a geometric random variable with parameter $q, q \in (0, 1)$. Moreover, assume that*

$$E\mid X_{n1} \mid^k = E\mid Z_1^{(m)} \mid^k, k = 1, 2, \ldots, r; r = 1, 2, \ldots. \tag{17}$$

Then, for $f \in C_B^r(\mathbb{R})$

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} \| = o(q^{r-1}), \quad as \quad q \to 0^+. \tag{18}$$

Proof. Since $f \in C_B^r(\mathbb{R})$ one has the Taylor series expansion up to r order. This yields, by virtue of assumption (17) and based upon the properties of Trotter' operator

$$\begin{aligned}
&\|T_{qS_{N_q}} f - T_{Z^{(m)}} \| \\
&\leq E(N_q) \, \|T_{qX_{n1}} f - T_{Z_1^{(m)}} f(x) \| \leq \frac{q^{r-1}}{r!} \int_0^{+\infty} \mid x \mid^r | f^{(r)}(\eta) - f(y) | dF_{X_{n1}}(x) \\
&\qquad + \frac{q^{r-1}}{r!} \int_0^{+\infty} \mid x \mid^r \mid f^{(r)}(\eta) - f(y) \mid dF_{Z_1^{(m)}}(x) \\
&\leq \frac{q^{r-1}}{r!} \int_0^{\frac{\delta}{q}} \mid x \mid^r \mid f^{(r)}(\eta) - f(y) \mid dF_{X_{n1}}(x) \\
&\qquad + \frac{q^{r-1}}{r!} \int_{\frac{\delta}{q}}^{+\infty} \mid x \mid^r \mid f^{(r)}(\eta) - f(y) \mid dF_{X_{n1}}(x) \\
&\qquad + \frac{q^{r-1}}{r!} \int_0^{\frac{\delta}{q}} \mid x \mid^r \mid f^{(r)}(\eta) - f(y) \mid dF_{Z_1^{(m)}}(x) \\
&\qquad + \frac{q^{r-1}}{r!} \int_{\frac{\delta}{q}}^{+\infty} \mid x \mid^r \mid f^{(r)}(\eta) - f(y) \mid dF_{Z_1^{(m)}}(x),
\end{aligned} \tag{19}$$

where $\delta > 0$, such that $\mid \eta - y \mid < q \mid x \mid < \delta$. Since $f \in C_B^r(\mathbb{R}), \forall \epsilon > 0 \exists \delta > 0$, such that $\mid f^{(r)}(\eta_2) - f^{(r)}(y) \mid < \epsilon$ if $x < \frac{\delta}{q}$ and $\mid f^{(r)}(\eta_2) - f^{(r)}(y) \mid < 2 \parallel f^{(r)} \parallel$ if $x > \frac{\delta}{q}$. Then, from inequalities of (19) we deduce that

$$\begin{aligned}
\| T_{qS_{N_q}} f - T_{Z^{(m)}} \| \leq \frac{q^{r-1}}{r!} \Big(& 2\epsilon E\mid X_{n1} \mid^r + 2\epsilon E\mid Z_1^{(m)} \mid^r + 2 \parallel f^r \parallel M_{X,r}(q) \\
& + 2 \parallel f^r \parallel M_{Z^{(m)},r}(q) \Big),
\end{aligned} \tag{20}$$

where
$M_{X,r}(q) = \int_{\frac{\delta}{q}}^{+\infty} \mid x \mid^r dF_{X_{n1}}(x) \to 0$ and $M_{Z^{(m)},r}(q) = \int_{\frac{\delta}{q}}^{+\infty} \mid x \mid^r dF_{Z_1^{(m)}}(x) \to 0$ as $q \to 0$ on account of finiteness of r-th absolute moments $E\mid X_{n1} \mid^r$ and

$E| X_{n1} |^r$. Then, it is follows that

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} \| = o\left(q^{r-1}\right) \quad \text{as} \quad q \to 0^+.$$

This completes the proof. $\square$

Theorem 3.3. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with mean $E(X_{n1}) = m < +\infty$ and finite variance $0 < D(X_{n1}) = \sigma^2 < +\infty, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Moreover, let $N_q, q \in (0,1)$ be a geometric variable with parameter $q, q \in (0,1)$, and suppose that N_q is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Then, for every $f \in C_B^1(\mathbb{R})$,*

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq 2\omega(f'; q)\left(m + \frac{1}{2}\sigma^2 + m^2\right). \tag{21}$$

In particular, suppose that $f' \in Lip(\alpha, M), 0 < \alpha \leq 1, 0 < M < +\infty$. Then

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq 2\left(m + \frac{1}{2}\sigma^2 + m^2\right) M q^{\alpha}. \tag{22}$$

Proof. By an argument analogous to that used for the proof of Theorem 3.2, with consideration of inequalities of Trotter operator

$$\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq E(N_q) \| T_{qX_{n1}} f - T_{Z_1^{(m)}} f \| = \frac{1}{q} \| T_{qX_{n1}} f - T_{Z_1^{(m)}} f \|.$$

Applying the operator $T_{qX_{n1}}$ to function $f \in C_B^1(\mathbb{R})$, with the Taylor series expansion, this yields

$$\begin{aligned} T_{qX_{n1}} f(y) &= \int_0^{+\infty} f(qx + y) dF_{X_{n1}}(x) \\ &= f(y) + qf' \int_0^{+\infty} x dF_{X_{n1}}(x) + q \int_0^{+\infty} x[f'(\eta_3) - f'(y)] dF_{X_{n1}}(x) \\ &= f(y) + qmf'(y) + q \int_0^{+\infty} x[f'(\eta_3) - f'(y)] dF_{X_{n1}}(x), \end{aligned} \tag{23}$$

where $| \eta_3 - y | < qx$. Analogously, applying the operator $T_{qZ_1^{(m)}}$ to function $f \in C_B^1(\mathbb{R})$, with the Taylor series expansion, we have

$$T_{qZ_1^{(m)}} f(y) = f(y) + qmf'(y) + q \int_0^{+\infty} x[f'(\eta_4) - f'(y)] dF_{X_{n1}}(x), \tag{24}$$

where $| \eta_4 - y |< qx$. Thus, combining both equations (23) and (24), using the properties of modulus of continuity, we get

$$
\begin{aligned}
&| T_{qX_{n1}} f(y) - T_{qZ_1^{(m)}} f(y) | \\
&\leq q \int_0^{+\infty} x \mid f^{'}(\eta_4) - f^{'}(y) \mid dF_{X_{n1}}(x) + q \int_0^{+\infty} x \mid f^{'}(\eta_4) - f^{'}(y) \mid dF_{Z_1^{(m)}}(x) \\
&\leq q[\omega(f^{'}; q) \int_0^{+\infty} x(1+x) dF_{Z_1^{(m)}}(x) + \omega(f^{'}; q) \int_0^{+\infty} x(1+x) dF_{X_{n1}}(x)] \\
&\leq 2q\omega(f^{'}; q) \left(m + \frac{1}{2}\sigma^2 + m^2 \right).
\end{aligned} \tag{25}
$$

We then have the desired estimation

$$
\begin{aligned}
\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| &\leq \frac{1}{q} \| T_{qX_{n1}} f - T_{Z_1^{(m)}} f \| \\
&\leq 2\omega(f^{'}; q) \left(m + \frac{1}{2}\sigma^2 + m^2 \right).
\end{aligned} \tag{26}
$$

Finally, for $f^{'} \in Lip(\alpha)$, we have

$$
\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq 2 \left(m + \frac{1}{2}\sigma^2 + m^2 \right) M q^{\alpha}.
$$

This completes the proof. □

Theorem 3.4. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with finite r-th absolute moment $E(| X_{nj} |^r) < +\infty, j = 1, 2, \ldots; r \geq 1$. Let $N_q, q \in (0, 1)$ be a geometric variable with parameter $q, q \in (0, 1)$, and suppose that N_q is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Moreover, assume that*

$$
E(| X_{nj} |^r) = E(| Z_j^{(m)} |^r); \quad r \geq 1. \tag{27}
$$

Then, for every $f \in C_B^{r-1}(\mathbb{R})$,

$$
\| T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq \frac{2q^{r-1}}{(r-1)!} \omega(f^{(r-1)}; q) \left(m^{r-1}(r-1)! + m^r r! \right), \tag{28}
$$

where $Z_j^{(m)}$ are independent exponential distributed random variables with common mean m, i.e. $Z_j^{(m)} \sim Exp(\frac{1}{m}), j = 1, 2, \ldots$

Proof. Our proof starts with the observation that

$$Z^{(m)} \stackrel{d}{=} q \sum_{j=1}^{N_q} Z_j^{(m)}.$$

and the Taylor series expansion for every $f \in C_B^{r-1}(\mathbb{R})$

$$f(x+y) = \sum_{j=0}^{r-1} \frac{f^{(j)}(y)x^j}{j!} + \frac{1}{(r-1)!} x^r \left(f^{(r-1)}(\eta_5) - f(y) \right),$$

where $\mid \eta_5 - y \mid < \mid x \mid$. Then, on account of assumption (27) and based on the properties of Trotter's operator, we have

$$\begin{aligned}
&\|T_{qS_{N_q}} f - T_{Z^{(m)}} f \| \leq E(N_q) \parallel T_{qX_{n1}} f(y) - T_{qZ_j^{(m)}} f(y) \parallel \\
&\leq \frac{1}{q} \parallel T_{qX_{n1}} f(y) - T_{qZ_j^{(m)}} f(y) \parallel \\
&\leq \frac{q^{r-1}}{(r-1)!} \int_0^{+\infty} \mid x \mid^{r-1} \mid f^{(r-1)}(\eta_5 - f^{(r-1)}(y)) \mid dF_{X_{n1}}(x) \\
&+ \frac{q^{r-1}}{(r-1)!} \int_0^{+\infty} \mid x \mid^{r-1} \mid f^{(r-1)}(\eta_5 - f^{(r-1)}(y)) \mid dF_{Z_1^{(m)}}(x) \\
&\leq \frac{q^{r-1}}{(r-1)!} \omega(f^{(r-1)}; q) \int_0^{+\infty} \mid x \mid^{r-1} (1 + \mid x \mid) dF_{X_{n1}}(x) \\
&+ \frac{q^{r-1}}{(r-1)!} \omega(f^{(r-1)}; q) \int_0^{+\infty} \mid x \mid^{r-1} (1 + \mid x \mid) dF_{Z_1^{(m)}}(x) \\
&\leq \frac{2q^{r-1}}{(r-1)!} \omega(f^{(r-1)}; q) \left(\int_0^{+\infty} x^{r-1} dF_{Z_1^{(m)}}(x) + \int_0^{+\infty} x^{r-1} dF_{Z_1^{(m)}}(x) \right) \\
&\leq \frac{2q^{r-1}}{(r-1)!} \omega(f^{(r-1)}; q) \left(m^{r-1}(r-1)! + m^r r! \right).
\end{aligned} \tag{29}$$

Thus, this completes the proof. □

Theorem 3.5. *Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and standard normal distributed random variables. Let $N_q, q \in (0,1)$ be a geometric variable with parameter q, $p \in (0,1)$, and suppose that N_q is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$. Then, for every $f \in C_B^2(\mathbb{R})$,*

$$\parallel T_{qS^2_{N_q}} f - T_{Z^{(1)}} f \parallel \leq \frac{q}{2} \parallel f^{''} \parallel \left(1 + 24\omega(f^{''}; q \right), \tag{30}$$

where $S^2_{N_q} = X^2_{n1} + X^2_{n2} + \ldots + X^2_{nN_q}$ and $Z^{(1)} \sim Exp(1)$.

Proof. We first observe that, for a exponential distributed random variable Z^1, we have

$$Z^{(1)} \stackrel{d}{=} q \sum_{j=1}^{N_q} Z_j^{(1)}. \tag{31}$$

Thus, based on properties of Trotter operator, we have

$$\begin{aligned} \| T_{qS_{N_q}^2} f - T_{Z^{(1)}} f \| &\leq E(N_q) \| T_{qX_{n1}^2} f - T_{qZ_1^{(1)}} \| \\ &= \frac{1}{q} \| T_{qX_{n1}^2} f - T_{qZ_1^{(1)}} \|, \end{aligned} \tag{32}$$

for every $f \in C_B^2(\mathbb{R})$. Then, based on the Trotter operator, for every function $f \in C_B^2(\mathbb{R})$ with the Taylor expansion, we have

$$\begin{aligned} &T_{qX_{n1}^2} f(y) = \int_0^{+\infty} f(qx+y) dF_{X_{n1}^2}(x) \\ &= f(y) + qy' \int_0^{+\infty} x dF_{X_{n1}^2}(x) + \frac{q^2}{2} f''(y) \int_0^{+\infty} x^2 dF_{X_{n1}^2}(x) \\ &\qquad + \frac{q^2}{2} f''(y) \int_0^{+\infty} x^2 \left(f''(\eta_9) - f''(y) \right) dF_{X_{n1}^2}(x) \\ &= f(y) + qy' + \frac{3q^2}{2} f''(y) + \frac{q^2}{2} f''(y) \int_0^{+\infty} x^2 \left(f''(\eta_6) - f''(y) \right) dF_{X_{n1}^2}(x), \end{aligned} \tag{33}$$

where $| \eta_6 - y | < q | x |$. By an argument analogous to the previous one, we get

$$\begin{aligned} &T_{qZ_1^{(1)}} f(y) = \int_0^{+\infty} f(qx+y) dF_{Z_1^{(1)}}(x) \\ &\quad = f(y) + qy' + q^2 f''(y) + \frac{q^2}{2} f''(y) \int_0^{+\infty} x^2 \left(f''(\eta_7) - f''(y) \right) dF_{Z_1^{(1)}}(x), \end{aligned} \tag{34}$$

where $\mid \eta_7 - y \mid < q \mid x \mid$. Combining (33) with (34) yields

$$\begin{aligned}
&\mid T_{qX^2_{n1}} f(y) - T_{qZ_1^{(1)}} f(y) \mid \\
&\leq \frac{q^2}{2} \parallel f^{''} \parallel + \frac{q^2}{2} \parallel f^{''} \parallel \int_0^{+\infty} x^2 \omega(f^{''}; qx) dF_{X^2_{n1}}(x) \\
&\qquad\qquad + \frac{q^2}{2} \parallel f^{''} \parallel \int_0^{+\infty} x^2 \omega(f^{''}; qx) dF_{Z_1^{(1)}}(x) \\
&\leq \frac{q^2}{2} \parallel f^{''} \parallel (1 + \omega(f^{''}; q) \int_0^{+\infty} x^2 (1+x) dF_{X^2_{n1}}(x) \\
&\qquad\qquad + \frac{q^2}{2} \parallel f^{''} \parallel (1 + \omega(f^{''}; q) \int_0^{+\infty} x^2 (1+x) dF_{Z_1^{(1)}}(x) \\
&\leq \frac{q^2}{2} \parallel f^{''} \parallel \left(1 + \omega(f^{''}; q) \left[9 + 8 \frac{\Gamma(3 + \frac{1}{2})}{\Gamma(\frac{1}{2})}\right]\right),
\end{aligned}$$

where $\Gamma(k) = \int_0^{+\infty} e^{-x} x^{k-1} dx, (k \geq 1)$, denotes the Gamma function with some particular values such that$\Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$. Then, using the computations of the Gamma function, it follows that

$$\begin{aligned}
&\parallel T_{qS^2_{N_q}} f - T_{Z^{(1)}} f \parallel \leq \frac{1}{q} \parallel T_{qS^2_{N_q}} f - T_{Z_1^{(1)}} f \parallel \\
&\leq \frac{q}{2} \parallel f^{''} \parallel \left(1 + \omega(f^{''}; q) \left[9 + 8 \frac{\Gamma(3 + \frac{1}{2})}{\Gamma(\frac{1}{2})}\right]\right) \\
&\leq \frac{q}{2} \parallel f^{''} \parallel \Big(1 + 24 \omega(f^{''}; q)\Big).
\end{aligned}$$

This completes the proof. □

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