Southeast Asian Journal of Sciences, Vol. 12, No 1 (2026), pp. 1-10
https://doi.org/10.36853/sajs0031

OPTIMIZATION OF TRUST PROPAGATION ON HILBERT SPACE IN COMPLEX NETWORKS

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Abstract

This paper introduces a Hilbert space formulation of trust propagation and develops an optimization framework for estimating global trust states in complex networks. Trust interactions are modeled using a bounded linear operator defined on the Hilbert space of node trust vectors. The global trust inference problem is formulated as a variational optimization problem whose solution represents the equilibrium trust configuration of the network. The proposed framework guarantees existence and uniqueness of optimal trust states under mild spectral conditions. An efficient iterative algorithm is derived using gradient-based optimization in Hilbert space. Preliminary experiments on simulated networks demonstrate the stability and convergence of the proposed approach compared with classical propagation models.

1 Introduction

Trust modeling has become a central problem in complex networks such as social media platforms, e-commerce systems, and peer-to-peer infrastructures [8].

Key words: hilbert space, embedding, operator, trust propagation, optimization.
2010 AMS Mathematics classification: 911D30, 91D10, 68U115, 68U35, 68M14, 68M115, 68T99.

Existing methods include EigenTrust [2], TrustRank [3], and PageRank-style propagation [1]. Although effective, these approaches are often heuristic and lack strong theoretical guarantees. Spectral graph theory provides powerful tools for analyzing propagation processes in networks [6]. However, relatively little work has explored trust propagation from a functional-analytic perspective.

More recent developments integrate machine learning techniques with network structure. Graph neural networks (GNNs) and geometric deep learning methods enable the learning of node representations directly from graph topology and attributes [10, 12]. Such approaches have significantly improved predictive performance in tasks such as link prediction, recommendation, and trust inference. However, many learning-based models emphasize empirical performance rather than providing rigorous analytical foundations for trust propagation dynamics.

From a theoretical perspective, the study of network processes can benefit from tools in spectral graph theory and functional analysis. Spectral methods provide insights into diffusion processes and stability properties of network operators [6, 7]. Hilbert space formulations, widely used in signal processing and dynamical systems, allow global states of networked systems to be represented as vectors in an infinite- or finite-dimensional functional space where operator-theoretic techniques can be applied [9]. Despite their potential, such functional-analytic perspectives have rarely been explored in trust modeling.

In this paper, we propose a Hilbert space formulation of trust propagation in complex networks. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a directed trust network with node set $\mathcal{V}$. We represent the global trust state as a vector in the Hilbert space $\ell^2(\mathcal{V})$ and model trust propagation using a bounded linear operator derived from the network adjacency structure. Within this setting, we formulate trust estimation as the minimization of a variational energy functional that captures the discrepancy between propagated and intrinsic trust signals. The proposed framework provides a mathematically rigorous perspective on trust propagation and allows the application of optimization methods in Hilbert spaces. Our main contributions are summarized as follows:

- We introduce a Hilbert space representation of trust states for complex networks.
- We define a trust propagation operator as a bounded linear operator acting on $\ell^2(\mathcal{V})$.
- We formulate global trust inference as a variational optimization problem in Hilbert space.
- We establish existence and uniqueness conditions for the optimal trust state and analyze the convergence of the proposed iterative algorithm.

The proposed operator-theoretic framework provides a foundation for future research directions including manifold-based trust representation, tensor-based multi-layer trust learning, and integration with geometric deep learning architectures.

2 Preliminaries

2.1 Network Representation

Definition 1. *A complex trust network is represented by a weighted directed graph*

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W), \tag{1}$$

where

- $\mathcal{V} = \{1, \ldots, n\}$ *is the set of nodes,*
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ *is the set of directed edges,*
- $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ *is the weighted adjacency matrix.*

An entry $w_{ij} \geq 0$ represents the level of trust that node i assigns to node j. If $(i, j) \notin \mathcal{E}$ then $w_{ij} = 0$. The matrix W therefore encodes the local trust relationships within the network.

2.2 Hilbert Space Representation

Denote

$$H = \ell^2(V) \cong \mathbb{R}^n \tag{2}$$

the Hilbert space of real-valued functions defined on the node set $\mathcal{V}$. A global trust configuration is represented by a vector

$$x = (x_1, \ldots, x_n) \in \mathcal{H}, \tag{3}$$

where x_i denotes the global trust score associated with node i. The Hilbert space $\mathcal{H}$ is equipped with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i, \tag{4}$$

which induces the norm

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{5}$$

This representation allows trust states to be analyzed using tools from functional analysis and operator theory.

2.3 Trust Propagation Operator

Trust propagation across the network is modeled by a linear operator induced by the network structure.

Definition 2 (Trust Propagation Operator)**.** *Let A be a normalized adjacency matrix associated with the graph $\mathcal{G}$. The trust propagation operator*

$$T : \mathcal{H} \to \mathcal{H} \tag{6}$$

is defined by

$$Tx = Ax, \tag{7}$$

for any trust vector $x \in \mathcal{H}$.

Proposition 1. *The trust propagation operator T is a bounded linear operator on $\mathcal{H}$.*

Proof. Since the network contains a finite number of nodes, the adjacency matrix $A \in \mathbb{R}^{n\times n}$ defines a linear operator on $\mathbb{R}^n$. All linear operators on finite-dimensional normed spaces are bounded. Therefore there exists a constant $C > 0$ such that

$$\|Ax\| \leq C\|x\| \tag{8}$$

for all $x \in H$. Hence the operator T is bounded. □

3 Optimization of Trust Propagation in Hilbert Space

Within the Hilbert space framework introduced in the previous section, the estimation of the global trust vector can be formulated as an optimization problem. The objective is to obtain a trust configuration that is consistent with network propagation while preserving prior trust information. Such variational formulations are commonly used in spectral graph learning and graph signal processing [6, 7].

3.1 Energy Functional

To estimate the global trust configuration, we define a regularized energy functional that balances trust consistency with the propagation operator and proximity to an initial trust estimate. Let $T : \mathcal{H} \to \mathcal{H}$ denote the trust propagation operator and let $x_0 \in \mathcal{H}$ be an initial trust vector derived from prior information or historical interactions. The following energy functional is defined as follows:

$$J(x) = \frac{1}{2}\|x - Tx\|^2 + \lambda\|x - x_0\|^2, \tag{9}$$

where $\lambda > 0$ is a regularization parameter. The first term enforces consistency with the network propagation mechanism. Minimizing $\|x - Tx\|^2$ encourages the trust vector to approach a stable fixed point of the propagation operator. The second term acts as a regularization component that restricts excessive deviation from the prior trust vector x_0. The trust estimation problem can therefore be expressed as the variational optimization problem

$$\min_{x \in \mathcal{H}} J(x). \tag{10}$$

3.2 Existence and Uniqueness

Proposition 2. *The optimization problem*

$$\min_{x \in \mathcal{H}} J(x) \tag{11}$$

admits a unique minimizer.

Proof. The functional $J(x)$ is a quadratic functional defined on the Hilbert space $\mathcal{H}$. Since $\lambda > 0$, the regularization term $\lambda\|x - x_0\|^2$ ensures that $J(x)$ is strictly convex. Furthermore, the functional is coercive, since

$$J(x) \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{12}$$

In finite-dimensional Hilbert spaces, every continuous strictly convex and coercive functional admits a unique minimizer; see, for example, standard results in convex optimization and functional analysis [9]. Hence the optimization problem possesses a unique optimal solution. □

3.3 Gradient-Based Optimization

To compute the optimal trust vector, we employ gradient-based optimization in the Hilbert space $\mathcal{H}$. Let I denote the identity operator on $\mathcal{H}$. The energy functional can be rewritten as

$$J(x) = \frac{1}{2}\|(I - T)x\|^2 + \lambda\|x - x_0\|^2. \tag{13}$$

Using standard calculus in Hilbert spaces [9], the gradient of $J(x)$ can be derived by differentiating the quadratic functional. This yields

$$\nabla J(x) = (I - T)^*(I - T)x + 2\lambda(x - x_0), \tag{14}$$

where $(I - T)^*$ denotes the adjoint operator of $(I - T)$. The optimal trust vector corresponds to the stationary point of the functional $J(x)$. Therefore, it satisfies the first-order optimality condition

$$\nabla J(x) = 0, \tag{15}$$

which leads to the equation

$$(I - T)^*(I - T)x + 2\lambda(x - x_0) = 0. \tag{16}$$

This equation characterizes the optimal trust configuration that balances the network propagation dynamics with the prior trust information.

3.4 Optimization Algorithm

The optimal trust vector can be approximated using an iterative gradient descent scheme. Starting from the initial trust vector $x^{(0)} = x_0$, the update rule is defined as

$$x^{(k+1)} = x^{(k)} - \eta \nabla J(x^{(k)}), \tag{17}$$

where $\eta > 0$ denotes the learning rate.

Algorithm 1 Optimization of Trust Propagation

Initialize $x^{(0)} = x_0$
for $k = 0$ to $K - 1$ **do**
  Compute propagated trust $Tx^{(k)}$
  Evaluate gradient

$$\nabla J(x^{(k)}) = (I - T)^*(I - T)x^{(k)} + 2\lambda(x^{(k)} - x_0)$$

  Update trust vector

$$x^{(k+1)} = x^{(k)} - \eta \nabla J(x^{(k)})$$

end for
return optimized trust vector $x^{(K)}$

This iterative scheme progressively minimizes the energy functional and converges to the optimal trust configuration under appropriate step-size conditions.

4 Experimental Illustration

To illustrate the proposed framework, we conduct experiments on simulated trust networks. Synthetic networks were generated using the Barabási–Albert scale-free model, which captures heterogeneous connectivity patterns commonly observed in social networks. Each network contains 200 nodes and approximately 800 directed edges. Initial trust values x_0 were randomly generated in the interval $[0, 1]$. The regularization parameter was set to $\lambda = 0.1$ and the gradient descent step size was $\eta = 0.01$. The optimization algorithm converges within 50–100 iterations for most simulated networks. The undirected scale-free network is then converted into a directed graph to represent asymmetric trust relationships. Let $A \in \mathbb{R}^{n\times n}$ denote the adjacency matrix of the resulting graph. Trust weights are assigned by sampling values from a uniform distribution on $[0, 1]$ for each existing edge and subsequently normalizing the outgoing weights of every node to obtain a row-stochastic matrix:

$$A_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}, \tag{18}$$

where w_{ij} denotes the sampled weight associated with edge (i, j).

The proposed Hilbert-space trust optimization method is compared with several baseline propagation approaches, including classical iterative trust propagation, PageRank [1], and EigenTrust [2]. Within the proposed framework, the global trust vector is obtained by minimizing the energy functional

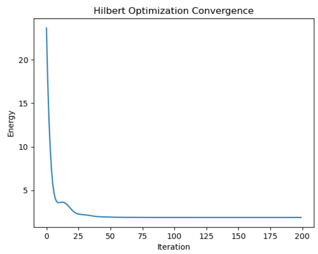
$$J(x) = \frac{1}{2}\|x - Ax\|^2 + \lambda\|x - x_0\|^2, \tag{19}$$

which balances propagation consistency with regularization toward a prior trust vector x_0. The methods are evaluated using the following quantitative metrics:

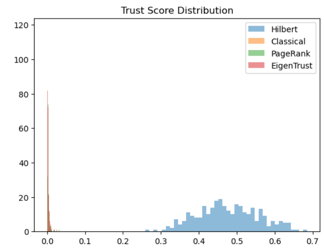
- optimization energy $J(x)$,
- spectral consistency $\|x - Ax\|/\|x\|$,
- Laplacian smoothness $x^\top Lx$,
- variance and entropy of the trust scores.

Figure 1(a) illustrates the convergence behavior of the proposed optimization procedure. The energy decreases rapidly during the early iterations and stabilizes after approximately 40 iterations, indicating efficient convergence of the gradient-based algorithm. Figure 1(b) presents the distribution of trust scores obtained by different methods. The Hilbert-space optimization produces a more heterogeneous distribution, whereas classical propagation approaches tend to yield near-uniform scores due to the stochastic structure of the propagation operator. Figure 1(c) shows the relationship between trust scores and node connectivity. A moderate correlation between node degree and inferred trust is observed, suggesting that the optimization captures meaningful structural information while avoiding trivial degree-based rankings. Figure 1(d) compares smoothness and entropy across the different methods. The Hilbert-based approach achieves lower smoothness values while maintaining higher entropy, indicating a stable yet diverse trust distribution across the network.

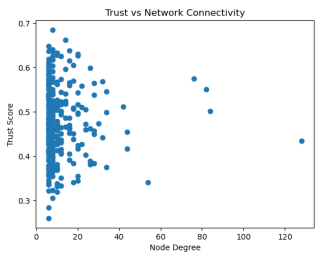
Overall, the experiments demonstrate that the proposed optimization framework produces stable trust estimates and mitigates degeneracy effects often observed in classical propagation models.



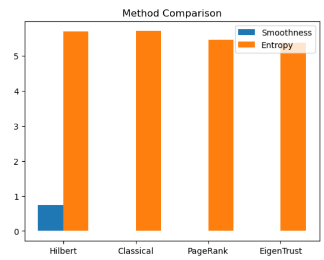
(a) Optimization convergence



(b) Trust score distribution



(c) Trust vs. connectivity



(d) Method comparison

Figure 1: Evaluation of the proposed Hilbert trust framework: (a) convergence of the optimization energy, (b) distribution of trust scores, (c) relationship between trust values and node connectivity, and (d) comparison of smoothness and entropy metrics.

5 Discussion

The Hilbert-space formulation enables the use of operator theory, convex optimization, and spectral analysis for studying trust propagation dynamics. First, representing trust states as elements of a Hilbert space allows the use of tools from functional analysis and operator theory to analyze trust dynamics on networks. This provides a principled mathematical foundation compared with purely heuristic propagation schemes. Second, the optimization formulation guarantees the existence and uniqueness of the trust solution under mild conditions on the propagation operator. This property enhances stability and interpretability, particularly in large-scale networks. Third, the framework supports efficient gradient-based algorithms for computing trust states in large graphs.

Finally, the Hilbert-space representation is flexible and can be extended to richer models of network interactions. Possible extensions include geometric trust embeddings, tensor-based multilayer representations, and integration with graph neural networks for data-driven trust learning.

6 Conclusion

This paper presented a Hilbert space optimization framework for trust propagation in complex networks. By modeling trust interactions as linear operators on the Hilbert space $\ell^2(\mathcal{V})$, the trust estimation problem was formulated as a convex optimization task with guaranteed existence and uniqueness of solutions. An iterative gradient-based algorithm was developed to compute the optimal trust vector efficiently.

Experimental results on simulated scale-free networks demonstrate the stability and effectiveness of the proposed approach compared with classical propagation methods.

The proposed framework establishes a mathematical foundation for future research at the intersection of functional analysis, spectral graph theory, and machine learning for trust modeling in complex networks.

References

[1] Page, L., Brin, S., Motwani, R., Winograd, T. The PageRank citation ranking: Bringing order to the web. Stanford InfoLab Technical Report, 1999.

[2] Kamvar, S. D., Schlosser, M. T., Garcia-Molina, H. The EigenTrust algorithm for reputation management in P2P networks. In: *Proceedings of the 12th International World Wide Web Conference (WWW)*, pp. 640–651, 2003.

[3] Gyöngyi, Z., Garcia-Molina, H., Pedersen, J. Combating web spam with TrustRank. In: *Proceedings of the 30th International Conference on Very Large Data Bases (VLDB)*, pp. 576–587, 2004.

[4] Golbeck, J. *Computing and Applying Trust in Web-Based Social Networks.* PhD Thesis, University of Maryland, 2009.

[5] Tran Dinh, Q., Pham Phuong, T. Treextrust: Topic-aware computational trust based on interaction experience, reputation of users with similarity and path algebra of graphs in social networks. *Journal of Computer Science, AGH*, 26(2), 2025.

[6] Chung, F. *Spectral Graph Theory.* CBMS Regional Conference Series in Mathematics, American Mathematical Society, 1997.

[7] Shuman, D. I., Narang, S. K., Frossard, P., Ortega, A., Vandergheynst, P. The emerging field of signal processing on graphs. *IEEE Signal Processing Magazine*, 30(3), 83–98, 2013.

[8] Newman, M. E. J. *Networks: An Introduction.* Oxford University Press, 2010.

[9] Rudin, W. *Functional Analysis.* 2nd Edition, McGraw–Hill, 1991.

[10] Bronstein, M. M., Bruna, J., LeCun, Y., Szlam, A., Vandergheynst, P. Geometric deep learning: Going beyond Euclidean data. *IEEE Signal Processing Magazine*, 34(4), 18–42, 2017.

[11] Kipf, T. N., Welling, M. Semi-supervised classification with graph convolutional networks. In: *International Conference on Learning Representations (ICLR)*, 2017.

[12] Hamilton, W. L. *Graph Representation Learning.* Morgan & Claypool Publishers, 2020.