Southeast-Asian J. of Sciences Vol. 2, No. 2 (2013) pp. 131-140

SOME EXAMPLES OF FINITE TYPE FRACTALS IN THREE DIMENSIONAL SPACES

Mai The Duy

University of Finance-Marketing (UFM)
Ho Chi Minh City, Viet- nam
e-mail: vnenlighten@yahoo.com

Abstract

By choosing the contraction functions in the Interated Function System we extend the construction from two dimensional spaces to three dimensional spaces to build self-similar sets in 3-spaces. We also extend the neighbor map concept for the attractors which have difference sizes of sub-pieces to make them became finite type. Some interesting examples of self similar sets in three dimensional space are given.

1 Introduction

A Fractal in general, is a rough or fragmented geometric shape that can be split into parts, each of which is a reduced-size copy of the whole. This essential property is called self-similarity. A fractal usually has Hausdorff dimension which is greater than its topological dimension. Now with the aid of computer programs, fractal geometry has recently grown and is continuing to grow and we can visualize the beauty of many of the images that they have discovered.

Self-similar sets are a class of fractals which can be rigorously defined and treated by mathematical methods. In 1981 Hutchinson rigorously defined self-similar sets by this equation

$$F = f_1(F) \cup f_2(F) \cup ... \cup f_m(F),$$

Key words: contraction functions, self similar sets, fractal geometry, Hausdorff dimension, Neighbor maps.

where f_i, $i = 1, ..., m$ are contracting maps (IFS) on $\mathbb{R}^d$. Hutchinson proved that for given maps there is exactly one compact non-empty set, F, which fulfils the equation. This set F is called fractal or attractor of IFS.

Neighbor maps, which can be considered as a representation of relative position of pairs of non-empty intersecting sub-pieces. They were first introduced by C.Bandt and S. Graf[3]. We extend the definition of Bandt on neighbor maps and neighbor graphs. Two sub-pieces are neighbors if they intersect each other and the relation between their sizes must be the same with the relation between the sizes of the pieces on the original fractal. We also extend fractal constructions from two-dimensional space to three-dimensional space. In natural, many things has the fractal structure such as fern leaf, cloud, mountains. We try to find the self similar or self affine structure of those things. From the geometric point of view, the interesting self-similar sets is the self-similar sets that have the sub-pieces just touching or have exact overlap. So we have to enlarge them to see whether they are Cantor sets or they have overlap. We using a very strong tool that is the neighbor maps to discover that fact.

2 Magnify fractals - The neighbor maps

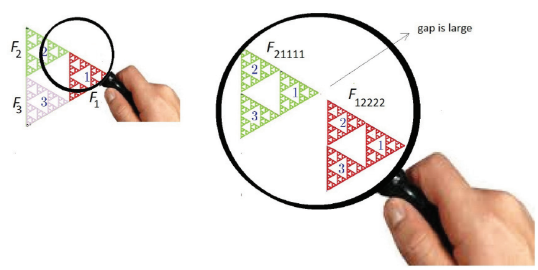

Figure 1: Using the neighbor maps we can magnify infinitely any self-similar sets

Fractals are sets or entities that look the same under magnification. Small pieces of such a set are similar to the whole set. Such sets are "self-similar."

To obtain a interesting structure in the self-similar sets, it is often required that overlaps between pieces are sufficiently thin or just touching, which is expressed by the open set condition:

Definition *We say that the IFS $\{f_1, ..., f_m\}$ satisfies the open set condition (OSC), if there exists an open set V such that*

$$\bigcup_{i=1}^{m} f_i(V) \subset V \text{ and } f_i(V) \cap f_j(V) = \emptyset, \ \forall i \neq j \in \{1, ..., m\}.$$

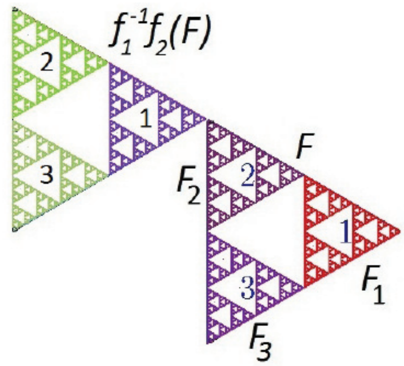

Figure 2: The relative position of sub-pieces.

We call such open set V a feasible open set of the f_i, or of F.

The OSC controls the overlap of the sub-pieces F_i of fractal F. If an IFS satisfies the open set condition then the Hausdorff dimension and the self-similarity dimension of the attractor coincide.

However, it is not easy to check the open set condition. In 1992 Bandt and Graf introduced an algebraic equivalent for OSC [3]. We take some notations, let $f_i : \mathbb{R}^d \to \mathbb{R}^d$ contractive similarities with contraction factor r, $i \in I = \{1, ..., m\}$ and $F = \bigcup_{i=1}^n f_i(F)$, $I^n := \{(u_i)_{i=1,...,n} \mid u_i \in I \ \forall i = 1, ..., n\}$, $I^* := \bigcup_{n=1}^\infty I^n$, for $u := u_1...u_n \in I^n$, define $f_u := f_{u_1} \circ ... \circ f_{u_n}$ and $F_u := f_u(F)$. Given an IFS $\{f_1, ..., f_m\}$, for each $u, v \in I^*$, $u = u_1u_2...$ and $v = v_1v_2...$, where $u_k,\ v_k \in I$, $k \in \mathbb{N}$. Let $\mathcal{N} = \{h = f_u^{-1} f_v \mid u, v \in I^*, u_1 \neq v_1\}$. The algebraic formulation of OSC reads as following Theorem

Theorem [3] *The iterated function system $\{f_1, ..., f_m\}$ satisfies the open set condition if and only if there exists $\delta > 0$ such that $\| h - id \| > \delta$, for all $h \in \mathcal{N}$.*

The norm in this theorem is the norm on affine maps, which can be $\| g \| := \| A \| + |b|$ if $g = Ax + b$, where

$$\| A \| = \max\{\| Ax \| \,|x \in \mathbb{R}^d \text{ with } \quad \| x \| \leq 1\}.$$

In 2001, Bandt [4] described an algorithm deciding on separation, when all the contraction factors are equal to r. The algorithm is as followed: Starting with identity map id, we applied the automorphism

$$h_{ij}(g) := f_i^{-1}.g.f_j, \quad i, j = 1, \cdots, m, \text{ and } i \neq j$$

Repeat this process with the obtained maps belonging to a neighborhood U of id until all the maps run out of U. The reality of this algorithm is confirmed by following proposition:

Proposition [4, Lemma 4.1] *Given similarities $f_i = rA_i(x + a_i)$, where $r \in (0, 1)$, and A_i are orthogonal matrices. Let U be the neighborhood of id in the space of similarities defined as*

$$U := \{sBx + b \mid |b| \leq \frac{(1+s)c}{1-r}\} \text{ where } c = \max_{i\in\{1,\cdots,m\}} |a_i|.$$

Then the complement of U is mapped into itself by each h_{ij}.

Neighbor maps, which can be considered as a representation of relative position of pairs of non-empty intersecting sub-pieces. They were first introduced by C.Bandt and S. Graf[4]. We take the sierpinski gasket to explain the neighbor maps. Suppose that our sirpinski gasket have three sub-pieces in the origin (see Figure 2) and the IFS have three contraction functions $\{f_1, f_2, f_3\}$. The relative position of sub-pieces F_1 and F_2 were represented by the map $g = f_1^{-1}f_2(F)$, The relative position between sub-pieces F_1 and F_2 are the same with the relative position between the fractal F and its image $f_1^{-1}f_2(F)$. Suppose that $F_1 \bigcap F_2$ have very small gap (they do not touching when the contraction factors r are smaller than 0.5, let take $r = 0.499$). Because the relative position between sub-pieces F_1 and F_2 are the same with the relative position between the fractal F and its image $f_1^{-1}f_2(F)$ then $F \bigcap f_1^{-1}f_2(F)$ also has a gap. Because F is larger than F_1 then the gap of $F \bigcap f_1^{-1}f_2(F)$ is bigger than the gap of $F_1 \bigcap F_2$. When we go to the next level; the relative position between F_{12} and F_{21} are the same with the relative position between the fractal F and $f_{12}^{-1}f_{21}(F)$. We continue other deeper level we have the relative position between F_{12222} and F_{21111} are the same with the relative position between the fractal F and $f_{12222}^{-1}f_{21111}(F)$. So we can imagine that we enlarge F_{12222} equal to F so the gap between F_{12222} and F_{21111} also to be enlarged that we can see in the Figure 1. It leads us to understand that when we apply the Bandt's algorithm this example then the algorithm will never stop because the gap go bigger and bigger, that out of the neighborhood U and never come back to h_{ij}. Bandt and Graf have give the definition of neighbor map for the maps have the same contraction factors. Given $f_1, ..., f_n$ a *neighbor map* is an element of the set $\{f_u^{-1}f_v \mid F_u \bigcap F_v \neq \emptyset, u \neq v \in I^n\}$.

We extend the definition of Bandt on neighbor maps and neighbor graphs. Two subpieces are neighbors if they intersect each other and the relation between their sizes must be the same with the relation between the sizes of the pieces on the original fractal. A type is a standardized relative position of two intersecting pieces of the fractal.

Definition: We say two pieces F_u and F_v are *neighbors* if

$$F_u \cap F_v \neq \emptyset, u, v \in I^*, u_1 \neq v_1, \min_{k,l\in I} \frac{r_k}{r_l} \leq r_{u,v} \leq \max_{k,l\in I} \frac{r_k}{r_l}.$$

where $u = u_1u_2...., v = v_1v_2....$ and $r_{u,v}$ is the contraction factor of the contracting map $f_u^{-1}f_v$, $r_{u,v} = \frac{r_v}{r_u}$.

With above definition two pieces F_u and F_v are neighbors if they intersect each other and their sizes can be comparable that they are not big different.

Definition: A neighbor map is an element of the set

$$V = \{f_u^{-1} f_v \mid F_u \cap F_v \neq \emptyset, F_u \text{ and } F_v \text{ are neighbors } \}$$

3 The neighbor graph-Finite type Fractals

The two pieces f_u and F_v are neighbors, and the type of that relation between f_u and F_v are presented by the neighbor map $h = f_u^{-1} f_v$. Every fractal that has a finite type neighbor map is called a finite type fractal. A type is a standardized relative position of two intersecting pieces of the fractal and the neighbor graph will show the relation between types. More detail we have

Definition: The neighbor graph $G = (V, E)$ of an IFS $\{f_i\}_{i=1}^m$ is given by the sets

$$V = \{f_u^{-1} f_v \mid F_u \cap F_v \neq \emptyset, F_u \text{ and } F_v \text{ are neighbors } \},$$
$$E = \{(g, h, ij) \in V \times V \times I^2 \mid h = f_i^{-1} g f_j \quad \min_{k,l \in I} \frac{r_k}{r_l} \leq r_h \leq \max_{k,l \in I} \frac{r_k}{r_l}\}$$

where $i, j \in I$, $r_h = \frac{r_j}{r_i} r_{u,v}$.

The fractals has finite type neighbors is finite fractals. If a self-similar set is a finite type fractal then the Bandt's algorithm will stop after sometime and we get the number of types. In the following part of this paper we give many new examples of finite type fractals in three dimensional space.

4 Some three-dimensional examples - The choice of IFS

Until now there are just a few examples of fractals in $\mathbb{R}^d$ with $d \geq 3$. We have two examples are well-known: the Menger sponge and the fractal tetrahedron [19] and recently, the three-dimensional twindragon [1] and the fractal octahedron, the three -dimensional modification of Sirpinski's triangle [6]. Rendering the pictures in this paper we use new solfware packages from Russia for three dimensional fractals [14, 15, 16] . So that the fractal objects in three dimensional space can be visualized. That could be one of the reasons make the research on three-dimensional self-similar fractals has been increasing.

In this paper we turn fractal in plane to fractal in space by two ways: The first way is we change the number of functions in IFS (Example 6), and the second way is to keep the number of subpices, it means keep the number of the functions in the IFS of both versions 2D and 3D (Examples 1 up to 5).

In plane in each function if IFS we usually using one rotation matrix, when we go to space we have to combine many rotation. In this paper we use three rotation matrices M,M',M'' where M is the rotation by $\frac{3\pi}{2}$ in the X-axis combine with the rotation by $\frac{3\pi}{2}$ in the Y-axis and M' is the rotation by $\frac{3\pi}{2}$ in the X-axis combine with the rotation by $\frac{\pi}{2}$ in the Y-axis, more exactly :

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, M' = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \text{and } M'' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Example 1. The simplest 3D fractal is the rectangular cuboid

$$F = [0, n] \times [0, \sqrt[3]{n}] \times [0, \sqrt[3]{n^2}]$$

with the IFS: $\{f_k(x) = rMx + (k-1)\}$, $k = 1, 2, ..., n$ and $r = 1/\sqrt[3]{n}$

Example 2 (See Figure 1). The rectangular cuboid canbe made by different sizes of subpieces:

$$F = [0, 2] \times [0\sqrt[3]{2}] \times [0, \sqrt[3]{4}]$$

The IFS has three functions:

$f_1(x) = rMx$, $f_2(x) = r^2M^2x + (1, 0, 0)'$ and $f_3(x) = r^2M''x + (1, \sqrt[3]{2}, \sqrt[3]{4})'$, $r = 1/\sqrt[3]{2}$

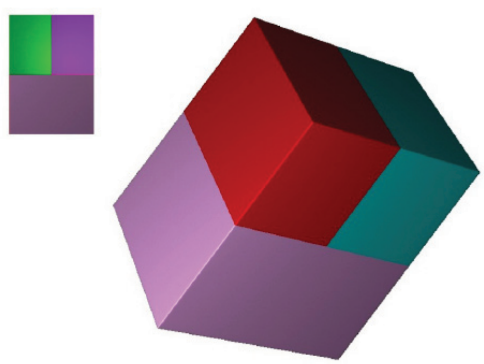

Figure 3: The rectangular cuboid in the Example 2

Example 3 (See Figure 4). The spiral fractals made of three equal sizes of subpieces. The IFS: $f_1(x) = rMx - v$,$f_2(x) = -rMx - v$, $f_3(x) = rMx$ where $r = 1/\sqrt[3]{3}$, $v = (1, 0, 0)'$. About the type of spiral fractals we can see the

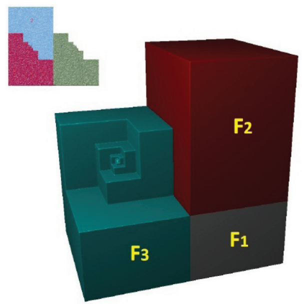

Figure 4: The spiral fractal

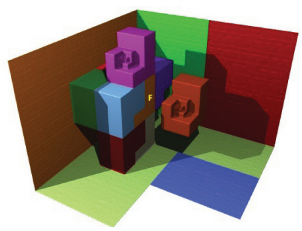

Figure 5: Eight face neighbor types of the spiral fractal

Figure 5 and count the numbers of sub-pieces around F. When neighboring pieces meeting in a single point or a line are neglected we can see 8 face neighbor types. Other versions of spiral fractals with different sizes of sub pieces can see in the Figure 6

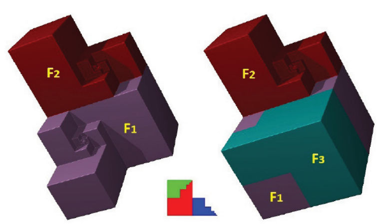

Figure 6: The spiral fractal with three different sub-pieces

Example 4 (See Figure 6). The spiral fractals made of three different sizes of subpieces. The IFS: $f_1(x) = rMx + v, f_2(x) = r^2M^2x,\ f_3(x) = -r^2M^2x + v$ where $r = 1/\sqrt[3]{2}$, $v = (-1/3,\ \sqrt[3]{2}/3,\ \sqrt[3]{4}/3)'$.

Example 5 (See Figure 7). The new Menger sponse uses only 4 contraction

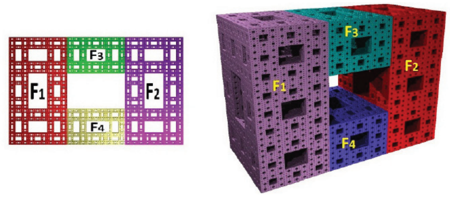

Figure 7: The new Menger sponse with 4 contraction functions in IFS

functions in The IFS: $f_1(x) = rMx + v, f_2(x) = rMx + (2,0,0)'$, $f_3(x) = r^2M^2x + (1,0,0)'$, $f_4(x) = r^2M^2x + (1,0,2/\sqrt[3]{3})'$ where $r = 1/\sqrt[3]{3}$.

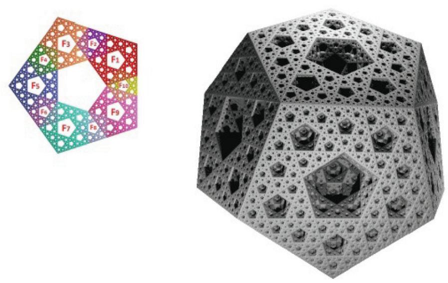

Figure 8: The golden dodecahedron fractal

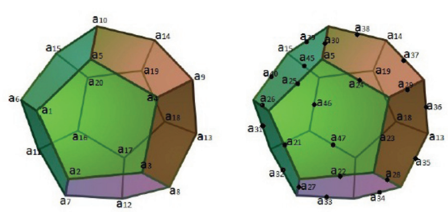

Figure 9: The 50 fixed points of the functions in IFS of the golden dodecahedron

Example 6 (See Figure 8). The golden dodecaheron fractal has 50 contraction functions in IFS $f_k(x) = \delta(x - a_k) + a_k$ if $k = 1, ..., 20$, and $f_k(x) = \delta^2(x - a_k) + a_k$, if $k = 21, ..., 50$, where δ is the golden ratio and a_k are the vertex points on the dodecahedron and the midpoints of the lines which connect that vertex points (see Figure 9).

When cutting the golden dodecahedron fractal we have slices as we can see in the Figure 10 and Figure 11. My friend, Ruediger suggests me that the

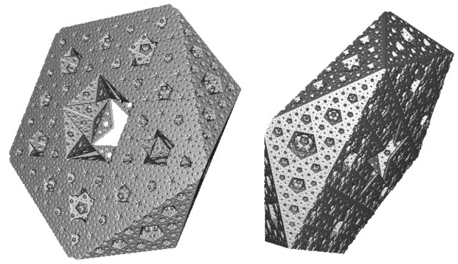

Figure 10: Two sides of cutting slides of the golden dodecahedron

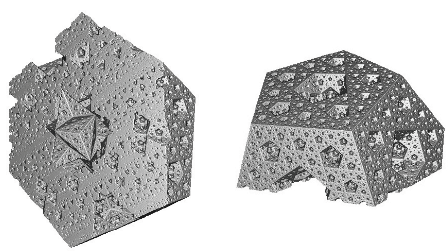

Figure 11: There is a hole in the center of the golden dodecahedron fractal.

golden dodecahdron fractal contains plane segments and in the center there is a hole. We can implies that holes exists almost everywhere, but it need tobe proved so there is much left to explore about this golen dodecahedron fractal.

References

[1] C. Bandt and M. Mesing , *Fractals of finite type*, Banach Center,Publications 84 (2009), 131-148.

[2] C. Bandt, *Self-Similar Sets 5. Integer Matrices and Fractal Tilings of Rn*, Proc. Amer. Math. Soc. 112 (1991), 549-562.

[3] C. Bandt and S. Graf, *Self-similar sets 7. A characterization of selfsimilar fractals with positive Hausdorff measure*, Proc. Amer. Math. Soc 114 (1992), 995-1001.

[4] C. Bandt, *Self-similar measures*, Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems (ed. B. Fiedler), Springer (2001), 31-46

[5] C. Bandt and N.V. Hung , *Fractal n-gons and their Mandelbrot sets*, Nonlinearity 21 (2008), 26532670.

[6] C.Bandt, M. Duy and M. Mesing, *Three-Dimensional Fractals*, Mathematical Intelligencer, Volume 32 (2010), Number 3, 12-18(web sites : http://www.springerlink.com/content/w200578285042037/)

[7] M. Duy, *Some self-similar constructions in two and three dimensions and their neighbor geometry*, dissertation, Greifswald university 2011 (web sites: http://ub-ed.ub.uni-greifswald.de/opus/volltexte/2011/993/)

[8] M.F. Barnsley , *Fractals Everywhere*, 2nd ed., Academic Press,1993.

[9] J.E. Hutchinson , *Fractals and self-similarity*, ndiana Univ. Math.J. 30 (1981), 713747.

[10] K.J. Falconer , "Fractal Geometry. Mathematical Foundations and Applications", Wiley 1990.

[11] H. Jones and A. Campa , *Fractals Based on Regular Polygons and Polyhedra*, N.M. Patrikalakis (ed.) Scientific Vizualization of Physical Phenomena, Springer, New York, 1991, 299-314.

[12] B.B. Mandelbrot, "The Fractal Geometry of Nature", Freeman, San Francisco, 1982.

[13] K.A. Roth, *Julia sets that are full of holes*, Math. Intelligencer 30, No. 4 (2008), 51-56.

[14] "Chaoscope software", Chaoscope software developed by N. Desprez,http://www.chaoscope.org/.

[15] "IFS Builder 3d software",created by Kravchenko Alexei and Mekhontsev Dmitriy, graduates of Novosibirsk State University (NSU), http://fractal.nsu.ru/builder3d.

[16] "Fractracer software",created by Mekhontsev Dmitriy, http://fractracer.com/

[17] R. Darst, J.A. Palagallo and T.E. Price, *Fractal tiling in the plane*, Mathematics Magagine, 71 (1988), 22-23.

[18] D. Broomhead, J. Montaldi and N. Sidorov, *Golden gaskets: variations on the Sierpinski sieve*, Nonlinearity 17 (2004), 1455-1480

[19] http://www.mathpaint.blogspot.com