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CHARACTERIZATIONS OF REGULAR ORDERED TERNARY SEMIGROUPS IN TERM OF FUZZY SUBSETS

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Abstract

In this paper, we show that the results in [6] are also valid on ordered ternary semigroups. Finally, we characterized regular ordered ternary semigroups by the properties of their fuzzy left [right, lateral] ideals, fuzzy quasi-ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals.

1 Introduction

The literature of a ternary algebraic system was introduced by D. H. Lehmer [13] in 1932. He investigated certain ternary algebraic system called triplexes which turn out to be ternary groups. The notion of ternary semigroup was known to S. Banach. He showed by an example that ternary semigroup does not necessarily reduced to an ordinary semigroup. In [15], M. L. Santiago developed the theory of ternary semigroups.

The concept of fuzzy sets was introduced by Zadeh [18] in 1965. Many researcher who are involved in studying, applying, refining and teaching fuzzy sets have successfully applied this theory in many different fields. The ideal

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theory in ternary semigroups was studied by F. M. Sioson [16] in 1965. In [3], T. K. Dutta, S. Kar and B. K. Maity studied some properties of regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups and characterized them by using various ideals of ternary semigroups. Fuzzy semigroups have been first considered by Kuroki [9, 10, 11].

Our aim in this paper is twofold.

1. To show that some results in [6] are also valid on ordered ternary semigroups.
2. To characterize regular ordered ternary semigroups by the properties of their fuzzy left [right, lateral] ideals, fuzzy quasi-ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals.

2 Preliminaries

In this section, we give some basic definitions and results, which are necessary for the subsequent sections.

A ternary groupoid is a non-empty set S together with the ternary operation.

Definition 1. *A ternary groupoid S is called ternary semigoup if S satisfies the associative law defined as*

$$[[abc]de] = [a[bcd]e] = [ab[cde]],$$

for all $a, b, c, d, e \in S$.

For simplicity, we shall write $[abc]$ as abc. Any semigroup can be reduced to a ternary semigroup. However, S. Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this examples.

Example 1. *Let S be a semigroups. Define a mapping $S \times S \times S \to S$ by $abc = ac$ for all $a, b, c \in S$. Then S is a ternary semigroup.*

Example 2. *Let S be the set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $a, b, c, d, e \in S$. Now, abc is equal to the usual product of rational numbers, then $abc \in S$ and $(abc)de = a(bcd)e = ab(cde)$. Then S is a ternary semigroup.*

Example 3. *Let S={-i,0,i}. Then S is a ternary semigroup under the multiplication over complex numbers while S is not a semigroup under complex number multiplication.*

These example shows that every semigroup is a ternary semigroup and ternary semigroup is a generalization of semigroups.

Definition 2. *A ternary semigroup [groupoid] S is called an ordered ternary semigroup [groupoid] if there is a partial order $\leq$ on S such that $a \leq b$ implies $acd \leq bcd$, $cad \leq cbd$ and $cda \leq cdb$, for all $a, b, c, d \in S$.*

Example 4. *Let S be the set of all 2×2 matrices over the set of positive integers. Then S is a ternary semigroup with respect to the usual matrix multiplication. Also S is a poset with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then S is an ordered ternary semigroups.*

Example 5. *Let S be the set of all integers of the form $4n + 2$, where n is a nonnegative integer. If abc is $a + b + c$ (usual sum of the integers) and $a \leq b$ means a is less than or equal to b for all $a, b, c \in S$, then S is an ordered ternary semigroup.*

Example 6. *Let S be the set of all isotone mapping on the poset P. Let $f, g, h \in S$. Denote by $f \circ g \circ h$ the usual composition mapping of f, g and h. Then S is a ternary semigroup. We define a relation $\leq$ on S by $f \leq g$ if $f(a) \leq g(a)$ for all $a \in P$. This relation is a partial order on S and as such S is a partial order set. Then S is an ordered ternary semigroup.*

Example 7. *For $a \in [0, 1]$, let $S = [0, a]$. Then S is an ordered ternary semigroup under usual multiplication and usual partial order relation.*

For any subset A of an ordered ternary semigroup S, we denote

$$(A] = \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

For $A, B, C \subseteq S$, we write,

$$ABC = \{abc \mid a \in A, b \in B \text{ and } c \in C\}.$$

If $A, B, C \in S$ then, we have $A \subseteq (A]$, $((A]] = (A]$, $(A](B](C] \subseteq (ABC]$, $((A](B](C]] = (ABC]$ and for $A \subseteq B$, we have $(A] \subseteq (B]$. A non-empty subset A of S is called a *ternary subsemigroup* of S if $AAA \subseteq A$.

A non-empty subset A of an ordered ternary semigroup S is called a *left* [right, lateral] *ideal* [4] of S if (i) $SSA \subseteq A$ $[ASS \subseteq A, SAS \subseteq A]$ and (ii) for all $a \in S$ and $b \in A$ if $a \leq b$, then $a \in A$. We call A call an *ideal* [14] if A is left, right and lateral ideal of S.

A non-empty subset B of an ordered ternary semigroup S is called a *bi-ideal* of S if (i) $BBB \subseteq B$, (ii) $BSBSB \subseteq B$ and (iii) for all $a \in B$ and $b \in S$ if $b \leq a$, then $b \in B$. A non-empty subset Q of S is called a *quasi-ideal* [1] of S if (i) $(QSS] \cap (SQS] \cap (SSQ] \subseteq Q$ and $(QSS] \cap (SSQSS] \cap (SSQ] \subseteq Q$ (ii) for all $a \in Q$ and $b \in S$ if $b \leq a$, then $b \in Q$.

Let X be a non-empty set. A mapping $f : X \rightarrow [0,1]$ is called a *fuzzy subset* of X. Let S be an ordered ternary semigroup and $a \in S$. Then we define

$$A_a := \{(x,y,z) \in S \times S \times S : a \leq xyz\}.$$

Let S be an ordered ternary semigroup and for three fuzzy subsets f, g and h of S, we define the multiplication $f \circ g \circ h$ of f, g and h as the fuzzy subset of S defined by:

$$(\forall a \in S)(f \circ g \circ h)(a) = \begin{cases} \sup\limits_{(u,v,w) \in A_a} \{\min\{f(u), g(v), h(w)\}\} & \text{if } A_a \neq \emptyset, \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

and in the set of all fuzzy subsets of S, $F(S)$, we define the order relation on $F(S)$ as follows: $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$. Finally for two fuzzy subsets f and g of S we define $f \cap g$ and $f \cup g$ as the fuzzy subsets of S defined by: For $x \in S$

$$(f \cap g)(x) = \min\{f(x), g(x)\} \text{ and } (f \cup g)(x) = \max\{f(x), g(x)\}.$$

For a subset A of S, we denote by f_A the characteristic function of A, that is, the mapping of S into $[0,1]$ defined by

$$f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in A$.

For an ordered ternary semigroup [groupoid] S, the fuzzy subset 1 of S is defined by $1(x) = 1$ for all $x \in S$.

Definition 3. *[2] Let S be an ordered ternary groupoid. A fuzzy subset f of S is called a fuzzy right [resp. left, lateral] ideal of S if for any $x, y, z \in S$,*

1. *$f(xyz) \geq f(x)$ [resp. $f(xyz) \geq f(z)$, $f(xyz) \geq f(y)$] and*
2. *$x \leq y$ implies $f(x) \geq f(y)$.*

We call f a *fuzzy ideal* of an ordered ternary semigroup S if f is a fuzzy right, left and lateral ideal of S.

Definition 4. *Let S be an ordered ternary groupoid. A fuzzy subset f of S is called a fuzzy quasi-ideal of S if*

1. *$(f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$,*
2. *$(f \circ 1 \circ 1) \cap (1 \circ 1 \circ f \circ 1 \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$ and*

3. *$x \leq y$ implies $f(x) \geq f(y)$ for every $x, y \in S$.*

Definition 5. *[17] Let S be an ordered ternary semigroup. A fuzzy subset f of S is called a fuzzy generalized bi-ideal of S if for any $u, v, w, x, y \in S$,*

1. *$f(uvwxy) \geq \min\{f(u), f(w), f(y)\}$ and*
2. *$x \leq y$ implies $f(x) \geq f(y)$.*

Definition 6. *[17] Let S be an ordered ternary semigroup. A fuzzy subset f of S is called a fuzzy ternary subsemigroup of S if for any $x, y, z \in S$, $f(xyz) \geq \min\{f(x), f(y), f(z)\}$.*

Definition 7. *[17] Let S be an ordered ternary semigroup. A fuzzy subset f of S is called a fuzzy bi-ideal of S if for any $u, v, w, x, y \in S$,*

1. *f is a fuzzy ternary subsemigroup of S,*
2. *$f(uvwxy) \geq \min\{f(u), f(w), f(y)\}$ and*
3. *$x \leq y$ implies $f(x) \geq f(y)$.*

An ordered ternary semigroup S is called *regular* [3] if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$, that is, an ordered ternary semigroup S is regular if for each $a \in S$, $a \in (aSa]$.

3 Fuzzy Ideals in Ordered Ternary Semigroups

The purpose of this section is to show that some results in [6] are also valid in ordered ternary semigroups.

Theorem 1. *If S is an ordered ternary groupoid, then the fuzzy right ideals of S are fuzzy quasi-ideals of S.*

Proof. Let f be a fuzzy right ideal of S. Let $x, y \in S$ such that $x \leq y$. Since f is a fuzzy right ideal of S, $x \leq y$ implies $f(x) \geq f(y)$. Let $a \in S$. If $A_a = \emptyset$, then we have $((f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f))(a) \leq f(a)$ and also $((f \circ 1 \circ 1) \cap (1 \circ 1 \circ f \circ 1 \circ 1) \cap (1 \circ 1 \circ f))(a) \leq f(a)$. If $A_a \neq \emptyset$, then

$$(f \circ 1 \circ 1)(a) = \sup_{(u,v,w) \in A_a} \{\min\{f(u), 1(v), 1(w)\}\}.$$

Let $(u, v, w) \in A_a$. Then $a \leq uvw$. Since f is a fuzzy right ideal of S, $f(a) \geq f(uvw) \geq f(u) \geq \min\{f(u), 1(v), 1(w)\}$. Thus $f(a)$ is an upper bound of $\{\min\{f(u), 1(v), 1(w)\} : (u, v, w) \in A_a\}$. It follows that $((f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f))(a) \leq f(a)$. Thus we have $(f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$. Similarly, we have $(f \circ 1 \circ 1) \cap (1 \circ 1 \circ f \circ 1 \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$. Therefore f is a fuzzy quasi-ideal of S.

In the same way, we can show that the fuzzy left [fuzzy lateral] ideals of an ordered ternary groupoid S are the fuzzy quasi-ideals of S.

Theorem 2. *If S is an ordered ternary semigroup, then the fuzzy quasi-ideals of S are fuzzy bi-ideals of S.*

Proof. Let f be a fuzzy quasi-ideal of S. Since f is a fuzzy quasi-ideal, $x \leq y$ implies $f(x) \geq f(y)$. Let $u, v, w, x, y, z \in S$. Now we have $f(xyz) \geq \min\{(f \circ 1 \circ 1)(xyz), (1 \circ f \circ 1)(xyz), (1 \circ 1 \circ f)(xyz)\}$. Since $(x, y, z) \in A_{xyz}$, $A_{xyz} \neq \emptyset$ and then

$$\begin{aligned}(f \circ 1 \circ 1)(xyz) &= \sup_{(r,s,t) \in A_{xyz}} \{\min\{f(r), 1(s), 1(t)\}\} \\ &\geq \min\{f(x), 1(y), 1(z)\} \\ &= f(x).\end{aligned}$$

Similarly, $(1 \circ f \circ 1)(xyz) \geq f(y)$ and $(1 \circ 1 \circ f)(xyz) \geq f(z)$. Hence $\min\{(f \circ 1 \circ 1)(xyz), (1 \circ f \circ 1)(xyz), (1 \circ 1 \circ f)(xyz)\} \geq \min\{f(x), f(y), f(z)\}$. Then $f(xyz) \geq \min\{f(x), f(y), f(z)\}$. This shows f is a fuzzy ternary subsemigroup. Since f is a fuzzy quasi-ideal of S, $f(uvwxy) \geq \min\{(f \circ 1 \circ 1)(uvwxy), (1 \circ 1 \circ f \circ 1 \circ 1)(uvwxy), (1 \circ 1 \circ f)(uvwxy)\}$. And we have $A_{uvwxy} \neq \emptyset$, since $(u, vwx, y) \in A_{uvwxy}$. Thus

$$\begin{aligned}(f \circ 1 \circ 1)(uvwxy) &= \sup_{(r,s,t) \in A_{uvwxy}} \{\min\{f(r), 1(s), 1(t)\}\} \\ &\geq \min\{f(u), 1(vwx), 1(y)\} \\ &= f(u).\end{aligned}$$

Similarly, $(1 \circ 1 \circ f \circ 1 \circ 1)(uvwxy) \geq f(w)$ and $(1 \circ 1 \circ f)(uvwxy) \geq f(y)$. Hence if we let $uvwxy = \alpha$, then $\min\{(f \circ 1 \circ 1)(\alpha), (1 \circ 1 \circ f \circ 1 \circ 1)(\alpha), (1 \circ 1 \circ f)(\alpha)\} \geq \min\{f(u), f(w), f(y)\}$. It follows that $f(uvwxy) \geq \min\{f(u), f(w), f(y)\}$. Altogether, f is a fuzzy bi-ideal of S.

Remark 1. *From the proof of Theorem 2, for any fuzzy subsets $f_1, \ldots f_5$ of S we have*

$$(f_1 \circ f_2 \circ f_3)(uvw) \geq \min\{f_1(u), f_2(v), f_3(w)\}$$

and also,

$$(f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5)(uvwxy) \geq \min\{f_1(u), f_2(v), f_3(w), f_4(x), f_5(y)\},$$

for any $u, v, w, x, y \in S$.

Theorem 3. *If S is a regular ordered ternary semigroup, then the fuzzy bi-ideals of S are fuzzy quasi-ideals of S.*

Proof. Let f be a fuzzy bi-ideal of S. Then for $x \leq y$ implies $f(x) \geq f(y)$. Let $x \in S$. Since S is a regular, $x \leq xax$ for some $a \in S$. This shows that $A_x \neq \emptyset$, so

$$\begin{aligned}(f \circ 1 \circ 1)(x) &= \sup_{(u,v,w) \in A_x} \{\min\{f(u), 1(v), 1(w)\}\},\\ (1 \circ f \circ 1)(x) &= \sup_{(u',v',w') \in A_x} \{\min\{1(u'), f(v'), 1(w')\}\} \text{ and}\\ (1 \circ 1 \circ f)(x) &= \sup_{(u'',v'',w'') \in A_x} \{\min\{1(u''), 1(v''), f(w'')\}\}.\end{aligned}$$

Let $(u, v, w), (u', v', w'), (u'', v'', w'') \in A_x$. Then $x \leq uvw, x \leq u'v'w'$ and $x \leq u''v''w''$. By compatibility of S, $x \leq xaxaxax$. Since $x \leq uvw$, $xaxaxax \leq uvwaxaxax$. It follows that $x \leq uvwaxaxax$. Since $x \leq u'v'w'$, $uvwaxaxax \leq uvwau'v'w'axax$. It follows that $x \leq uvwau'v'w'axax$. Since $x \leq u''v''w''$, $uvwau'v'w'axax \leq uvwau'v'w'axau''v''w''$.
It follows that $x \leq u(vwau'v')w'(axau''v'')w''$, or $x \leq us_1w's_2w''$ for some $s_1, s_2 \in S$. Then $f(x) \geq f(us_1w's_2w'') \geq \min\{f(u), f(w'), f(w'')\}$. If $f(u) \geq f(w') \geq f(w'')$, then $f(x) \geq f(w'')$. Then $f(x) \geq \min\{1(u''), 1(v''), f(w'')\}$. Hence $f(x)$ is an upper bound of $\{\min\{1(r), 1(s), f(t)\} : (r, s, t) \in A_x\}$. Thus $f(x) \geq (1 \circ 1 \circ f)(x)$. It follows that $f(x) \geq (1 \circ 1 \circ f)(x) \geq ((f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f))(x)$. In the cases of $f(u) \geq f(w'') \geq f(w'), f(w') \geq f(w'') \geq f(u), f(w') \geq f(u) \geq f(w''), f(w'') \geq f(w') \geq f(u)$ and $f(w'') \geq f(u) \geq f(w')$ are similarly. So we have $(f \circ 1 \circ 1) \cap (1 \circ f \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$. To show $(f \circ 1 \circ 1) \cap (1 \circ 1 \circ f \circ 1 \circ 1) \cap (1 \circ 1 \circ f) \subseteq f$ is similarly. Thus f is a fuzzy quasi-ideal of S.

Corollary 1. *Let S be a regular ordered ternary semigroup. Then the fuzzy quasi-ideals and the fuzzy bi-ideals are coincide.*

4 Regular Ordered Ternary Semigroups

In this section we characterized regular ordered ternary semigroups by the properties of their fuzzy left [right, lateral] ideals, fuzzy quasi-ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals.

Lemma 1. *Let S be an ordered ternary semigroup. Then we have $f \circ 1 \circ 1 \subseteq f, 1 \circ g \circ 1 \subseteq g$ and $1 \circ 1 \circ h \subseteq h$ for each fuzzy right ideal f of S, fuzzy lateral ideal g of S and fuzzy left ideal h of S.*

Proof. Let $a \in S$. If $A_a = \emptyset$, then $(f \circ 1 \circ 1)(a) = 0 \leq f(a)$. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}(f \circ 1 \circ 1)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f(u), 1(v), 1(w)\}\} \\ &= \sup_{(u,v,w)\in A_a} \{\min\{f(u)\}\} \\ &\leq \sup_{(u,v,w)\in A_a} \{\min\{f(a)\}\} = f(a).\end{aligned}$$

Thus we have $f \circ 1 \circ 1 \subseteq f$. For $1 \circ g \circ 1 \subseteq g$ and $1 \circ 1 \circ h \subseteq h$, the proof is similarly.

Lemma 2. *Let f, g and h be a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of an ordered ternary semigroup S, respectively. Then $f \cap g \cap h$ is a fuzzy quasi-ideal of S.*

Proof. Let $x, y \in S$ such that $x \leq y$. By the definition of f, g and h, $f(x) \geq f(y), g(x) \geq g(y)$ and $h(x) \geq h(y)$. Then we have $(f \cap g \cap h)(x) \geq (f \cap g \cap h)(y)$. Now, we let $F = f \cap g \cap h$. Let $a \in S$. Then

$$\begin{aligned}F(a) &= \min\{f(a), g(a), h(a)\} \\ &\geq \min\{(f \circ 1 \circ 1)(a), (1 \circ g \circ 1)(a), (1 \circ 1 \circ h)(a)\} \\ &\geq \min\{(F \circ 1 \circ 1)(a), (1 \circ F \circ 1)(a), (1 \circ 1 \circ F)(a)\} \\ &= ((F \circ 1 \circ 1) \cap (1 \circ F \circ 1) \cap (1 \circ 1 \circ F))(a).\end{aligned}$$

Hence $(F \circ 1 \circ 1) \cap (1 \circ F \circ 1) \cap (1 \circ 1 \circ F) \subseteq F$. Similarly, we have $(F \circ 1 \circ 1) \cap (1 \circ 1 \circ F \circ 1 \circ 1) \cap (1 \circ 1 \circ F) \subseteq F$. Therefore F is a quasi-ideal of S.

Lemma 3. *Let f, g and h be a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of an ordered ternary semigroup S, respectively. Then $f \cap g \cap h$ is a fuzzy generalized bi-ideal of S.*

Proof. Let $F = f \cap g \cap h$. Let $x, y \in S$ such that $x \leq y$. By the definition of f, g and h, $f(x) \geq f(y), g(x) \geq g(y)$ and $h(x) \geq h(y)$. Then we have $F(x) \geq F(y)$. Let $u, v, w, x, y \in S$. We have $f(uvwxy) \geq f(u), g(uvwxy) \geq g(vwx) \geq g(w)$ and $h(uvwxy) \geq h(y)$. Thus $F(uvwxy) \geq \min\{f(u), g(w), h(y)\} \geq \min\{F(u), F(w), F(y)\}$. Therefore $f \cap g \cap h$ is a generalized bi-ideal of S.

Lemma 4. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*
(1) S is regular.
(2) $R \cap M \cap L = (RML]$ for every right ideal R, lateral ideal M and left ideal L of S.

Proof. (1) $\Rightarrow$ (2) : Let R, M and L be a right ideal, a lateral ideal and a left ideal of S, respectively. Let $x \in (RML]$. Then $x \leq rml$ for some $r \in R, m \in M$ and $l \in L$. Thus $x \leq rml \in R \cap M \cap L$. This shows $(RML] \subseteq R \cap M \cap L$. Let $a \in R \cap M \cap L$. Since S is a regular, $a \leq axa$ for some $x \in S$. It follows that $a \leq (axa)(xax)(axa) \in RML$. Hence $R \cap M \cap L \subseteq (RML]$.
(2) $\Rightarrow$ (1) : Let $a \in S$. Now, we have right ideal generated by a, lateral ideal generated by a and left ideal generated by a is a right ideal, lateral ideal and a left ideal of S, respectively. Then $a \in ((aSS \cup a](SaS \cup SSaSS \cup a](SSa \cup a]] \subseteq (aSa]$. This shows S is a regular.

In general, S is regular if and only if $R \cap S \cap L = (RSL]$ for every right ideal R and left ideal L of S.

Proposition 1. *Let S be an ordered ternary semigroup and $\emptyset \neq R \subseteq S$. Then R is a right [resp., left, lateral] ideal of S if and only if f_R is a fuzzy right [resp., left, lateral] ideal of S.*

Theorem 4. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*
(1) S is regular.
(2) $f \cap g \cap h = f \circ g \circ h$ for every fuzzy right ideal f, fuzzy lateral ideal g and fuzzy left ideal h of S.

Proof. (1) $\Rightarrow$ (2) : Let f, g and h be a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of S, respectively. Let $a \in S$. Since S is a regular, $A_a \neq \emptyset$, that is $(a, xax, a) \in A_a$. Then

$$\begin{aligned}(f \circ g \circ h)(a) &= \sup_{(u,v,w) \in A_a} \{\min\{f(u), g(v), h(w)\}\} \\ &\geq \min\{f(a), g(xax).h(a)\} \\ &\geq \min\{f(a), g(a), h(a)\} \\ &= (f \cap g \cap h)(a).\end{aligned}$$

Thus $f \cap g \cap h \subseteq f \circ g \circ h$. Since $f \circ g \circ h \subseteq f \circ 1 \circ 1, f \circ g \circ h \subseteq 1 \circ g \circ 1$ and $f \circ g \circ h \subseteq 1 \circ 1 \circ h$, by Lemma 1, $f \circ g \circ h \subseteq f, f \circ g \circ h \subseteq g$ and $f \circ g \circ h \subseteq h$. Thus $f \circ g \circ h \subseteq f \cap g \cap h$. Therefore $f \circ g \circ h = f \cap g \cap h$.
(2) $\Rightarrow$ (1) : Let R, M and L be a right ideal, a lateral ideal and a left ideal of S, respectively. Then we have f_R, f_M and f_L is a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of S, respectively. Let $a \in R \cap M \cap L$. Since $a \in R \cap M \cap L$, we have $(f_R \cap f_M \cap f_L)(a) = \min\{f_R(a), f_M(a), f_L(a)\} = 1$ which implies $(f_R \circ f_M \circ f_L)(a) = 1$, that is $A_a \neq \emptyset$. Thus

$$1 = \sup_{(u,v,w) \in A_a} \{\min\{f_R(u), f_M(v), f_L(w)\}\}.$$

That is $f_R(u) = 1, f_M(v) = 1$ and $f_L(w) = 1$. It follows that $u \in R, v \in M$ and $w \in L$. Then $a \leq uvw \in RML$. Thus $a \in (RML]$. It is clear that $(RML] \subseteq R \cap M \cap L$. By Lemma 4, S is a regular.

Theorem 5. *For an ordered ternary semigroup S, the following assertions are equivalent:*
(1) *S is regular.*
(2) *$f \cap g = f \circ 1 \circ g$ for every fuzzy right ideal f and every left ideal g of S.*

Proof. (1) $\Rightarrow$ (2) : Let f and g be a fuzzy right ideal and a fuzzy left ideal of S, respectively. Let $a \in S$. Then $a \leq axa$ for some $x \in S$. It follows that $A_a \neq \emptyset$. Hence

$$\begin{aligned}(f \circ 1 \circ g)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f(u), 1(v), g(w)\}\} \\ &\geq \min\{f(a), 1(x), g(a)\} \\ &= (f \cap g)(a).\end{aligned}$$

Thus $f \cap g \subseteq f \circ 1 \circ g$. By Lemma 1, $f \circ 1 \circ g \subseteq f$ and $f \circ 1 \circ g \subseteq g$. Thus $f \circ 1 \circ g \subseteq f \cap g$. Therefore $f \cap g = f \circ 1 \circ g$.
(2) $\Rightarrow$ (1) : Let R and L be a right ideal and a left ideal of S, respectively. Then f_R and f_L be a fuzzy right ideal and a fuzzy left ideal, respectively. Let $a \in R \cap S \cap L$. Since $a \in R \cap S \cap L$, $(f_R \cap f_L)(a) = \min\{f_R(a), f_L(a)\} = 1$ implies $(f_R \circ 1 \circ f_L)(a) = 1$. That is $A_a \neq \emptyset$. Thus

$$1 = \sup_{(u,v,w)\in A_a} \{\min\{f_R(u), 1(v), f_L(w)\}\},$$

so $u \in R$ and $w \in L$. Thus $a \leq uvw \in RSL$, that is $a \in (RSL]$. It is clear that $(RML] \subseteq R \cap M \cap L$. Therefore S is regular by Lemma 4.

A fuzzy subset f of an ordered ternary semigroup S is called idempotent if $f \circ f \circ f = f$.

Theorem 6. *An ordered ternary semigroup in which all fuzzy generalized bi-ideals are idempotent is regular.*

Proof. Let f, g and h be an fuzzy right, an fuzzy lateral and an fuzzy left ideal of S, respectively. By Lemma 3, $F = f \cap g \cap h$ is a fuzzy generalized bi-ideal of S. By assumption, $F = F \circ F \circ F \subseteq f \circ g \circ h$. Now we have $f \circ g \circ h \subseteq f \circ 1 \circ 1$, $f \circ g \circ h \subseteq 1 \circ g \circ 1$ and $f \circ g \circ h \subseteq 1 \circ 1 \circ h$, thus by Lemma 1, $f \circ g \circ h \subseteq f$, $f \circ g \circ h \subseteq g$ and $f \circ g \circ h \subseteq h$. Hence $f \circ g \circ h \subseteq f \cap g \cap h$. It follows that $f \circ g \circ h = f \cap g \cap h$. By Theorem 4, S is a regular.

Corollary 2. *An ordered ternary semigroup in which all fuzzy bi-ideals are idempotent is regular.*

Corollary 3. *An ordered ternary semigroup in which all fuzzy quasi-ideals are idempotent is regular.*

Theorem 7. *For an ordered ternary semigroup S, the following assertions are equivalent:*
(1) *S is regular.*
(2) *$f \cap g \subseteq f \circ 1 \circ g$ for every fuzzy generalized bi-ideal f and every fuzzy left ideal g.*
(3) *$f \cap g \subseteq f \circ 1 \circ g$ for every fuzzy bi-ideal f and every fuzzy left ideal g.*
(4) *$f \cap g \subseteq f \circ 1 \circ g$ for every fuzzy quasi-ideal f and every fuzzy left ideal g.*

Proof. $(1) \Rightarrow (2)$: Let f and g be a fuzzy generalized bi-ideal and a fuzzy left ideal of S, respectively. Let $a \in S$. Then $a \leq axa$ for some $x \in S$. Thus

$$\begin{aligned}(f \circ 1 \circ g)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f(u), 1(v), g(w)\}\} \\ &\geq \min\{f(a), g(a)\} \\ &= (f \cap g)(a).\end{aligned}$$

$(2) \Rightarrow (3) \Rightarrow (4)$: It is routine since every fuzzy bi-ideal is fuzzy generalized bi-ideal and every fuzzy quasi-ideal is fuzzy bi-ideal of S.
$(4) \Rightarrow (1)$: Let f and g be a fuzzy right ideal and a fuzzy left ideal of S, respectively. Then we have f is a fuzzy quasi-ideal of S. By hypothesis, $f \cap g \subseteq f \circ 1 \circ g$ and by Lemma 1, $f \circ 1 \circ g \subseteq f \cap g$. Thus by Theorem 5, S is a regular.

Theorem 8. *For an ordered ternary semigroup S, the following assertions are equivalent:*
(1) *S is regular.*
(2) *$f \cap g \subseteq g \circ 1 \circ f$ for every fuzzy generalized bi-ideal f and every fuzzy right ideal g.*
(3) *$f \cap g \subseteq g \circ 1 \circ f$ for every fuzzy bi-ideal f and every fuzzy right ideal g.*
(4) *$f \cap g \subseteq g \circ 1 \circ f$ for every fuzzy quasi-ideal f and every fuzzy right ideal g.*

Proof. $(1) \Rightarrow (2)$: Let f and g be a fuzzy generalized bi-ideal and a fuzzy right ideal of S, respectively. Let $a \in S$. Then $a \leq axa$ for some $x \in S$. Thus

$$\begin{aligned}(g \circ 1 \circ f)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{g(u), 1(v), f(w)\}\} \\ &\geq \min\{f(a), g(a)\} \\ &= (f \cap g)(a).\end{aligned}$$

$(2) \Rightarrow (3) \Rightarrow (4)$: It is routine since every fuzzy bi-ideal is fuzzy generalized bi-ideal and every fuzzy quasi-ideal is fuzzy bi-ideal of S.
$(4) \Rightarrow (1)$: Let f and g be a fuzzy left ideal and a fuzzy right ideal of S, respectively. Then we have f is a fuzzy quasi-ideal of S. By hypothesis, $f \cap g \subseteq g \circ 1 \circ f$ and by Lemma 1, $g \circ 1 \circ f \subseteq f \cap g$. Thus by Theorem 4, S is a regular.

Theorem 9. *For an ordered ternary semigroup S, the following assertions are equivalent:*
(1) *S is regular.*
(2) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy generalized bi-ideal f and g.*
(3) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy bi-ideal f and g.*
(4) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy bi-ideal f and every fuzzy quasi-ideal g.*
(5) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy bi-ideal f and every fuzzy left ideal g.*
(6) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy quasi-ideal f and every fuzzy left ideal g.*
(7) *$f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$ for every fuzzy right ideal f and every fuzzy left ideal g.*

Proof. (1) $\Rightarrow$ (2) : Let f and g be fuzzy generalized bi-ideals of S. Let $a \in S$. Then $a \leq axa$ for some $x \in S$. Thus

$$\begin{aligned}(f \circ 1 \circ g)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f(u), 1(v), g(w)\}\} \\ &\geq \min\{f(a), 1(x), g(a)\} \\ &= (f \cap g)(a),\end{aligned}$$

and similarly, $(g \circ 1 \circ f)(a) \geq (f \cap g)(a)$. Therefore $f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f)$.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) : It is clear.
(7) $\Rightarrow$ (1) : Let f and g be a fuzzy right ideal and a fuzzy left ideal of S, respectively. Then we have $f \cap g \subseteq (f \circ 1 \circ g) \cap (g \circ 1 \circ f) \subseteq f \circ 1 \circ g$. And by Lemma 1, $f \circ 1 \circ g \subseteq f \cap g$. By Theorem 5, we have S is a regular.

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