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ON A MECHANICAL APPROACH TO A CLASS OF NON-AUTONOMOUS LINEAR SECOND ORDER STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

A class of non-autonomous linear second order stochastic differential equations is investigated by a technique based on mechanical approach. The system response is separated into the deterministic and random parts governed by two uncoupled differential equations which can be solved exactly. Mean-square responses of the system are compared with results obtained by the stochastic averaging method and Monte Carlo simulation.

1 Introduction

It is noted that no real system is exactly linear. Because the treatment of non-linear equations, however, is a good deal more complicated than the treatment of linear ones, many engineering systems can be modeled, to a first approximation, in terms of linear differential equations of motion, if the amplitude of

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motion is relatively small. The adoption of a linear model, however, is a quite desirable step in the analysis of any particular system under investigation. In this paper, we use linear second order stochastic differential equations to illustrate a mechanically approximate approach although such equations were studied in detail by many authors (e.g. see [1,2], and their earlier references). The purpose of the present paper is to apply a technique used in vibration analysis [3], which was first used to study the stochastic Duffing's equation, to find the exact solution of a non-autonomous linear second order stochastic differential equation. The concept of the technique comes from works of Caughey [4] and Piszczek [5], but in general form. The mean-square response of the system is compared with the numerical result obtained by the method of stochastic averaging and Monte Carlo simulation. The accuracy and reliability of the approach is validated by these comparisons. This technique can be applied to non-linear stochastic differential equations under investigation with two famous methods: averaging method and equivalent linearization method.
The structure of the paper's content is as follows. In Section 2, we present a mechanically analytical technique to the non-autonomous linear second order stochastic differential equation. Numerical results are in Section 3. And the final is summary and conclusions.

2 Analytical technique

Let's consider the non-autonomous linear stochastic differential equation in the form of

$$\ddot{x} + \omega^2 x = \varepsilon \left(-2\alpha \dot{x} + P \cos \nu t\right) + \sqrt{\varepsilon}\sigma\xi\left(t\right) \tag{1}$$

where α, P, σ, ω, ν are positive constants, and $\xi\left(t\right)$ is a white noise with unit intensity. To obtain the solution of Eq.(1) in mechanical approach, Eq.(1) is separated into two different kinds: a deterministic equation and a stochastic equation which can be solved exactly. To do that, the random response of Eq.(1) is assumed to include deterministic and random parts which can be separated completely. Thus, let us introduce two new variables m and u such that

$$x\left(t\right) = m\left(t\right) + u\left(t\right), \tag{2}$$

where m denotes the mathematical expectation of the response x, u denotes the random part of the response x which has the expectation to be zero

$$m = \langle x \rangle\,, \quad \langle u \rangle = 0, \tag{3}$$

where notation $\langle . \rangle$ denotes mathematical expectation operator. Substituting (2) into Eq. (1) yields

$$\left(\ddot{m} + 2\varepsilon\alpha\dot{m} + \omega^2 m\right) + \left(\ddot{u} + 2\varepsilon\alpha\dot{u} + \omega^2 u\right) = \varepsilon P \cos \nu t + \sqrt{\varepsilon}\sigma\xi\left(t\right). \tag{4}$$

By taking mathematical expectation on both sides of Eq. (4) and noting Eq. (3), one obtains a nonlinear equation for the deterministic part of the response where the right hand side is a purely periodic excitation with frequency ν

$$\ddot{m} + 2\varepsilon\alpha\dot{m} + \omega^2 m = \varepsilon P\cos\nu t. \tag{5}$$

Multiply Eq. (5) by -1 and add the obtained result into Eq. (4) to get the following stochastic differential equation

$$\ddot{u} + 2\varepsilon\alpha\dot{u} + \omega^2 u = \sqrt{\varepsilon}\sigma\xi\left(t\right). \tag{6}$$

Because Eq. (5) is a linearly damped system under harmonic force, its response takes the form [9]

$$m\left(t\right) = e^{-\varepsilon\alpha t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \frac{\varepsilon P}{\left(\omega^2-\nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2}\left[\left(\omega^2-\nu^2\right)\cos\nu t + 2\varepsilon h\nu\sin\nu t\right], \tag{7}$$

where A, B are constants, $\omega_d = \sqrt{\omega^2 - \left(\varepsilon\alpha\right)^2}$, $0 < \frac{\varepsilon\alpha}{\omega} < 1$. When t gets large the solution (7) approaches to the following

$$m\left(t\right) = \frac{\varepsilon P}{\left(\omega^2-\nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2}\left[\left(\omega^2-\nu^2\right)\cos\nu t + 2\varepsilon\alpha\nu\sin\nu t\right]. \tag{8}$$

On the other hand, because Eq. (6) is a autonomous linear stochastic differential equation, the impulse response function of the response of Eq. (6) takes the form [10]

$$h\left(t\right) = \frac{1}{\omega_d}e^{-\varepsilon\alpha t}\sin\omega_d t,\ t \geqslant 0. \tag{9}$$

According to theory of time domain vibration analysis, the response of Eq. (6) can be found in the following form

$$u\left(t\right) = \sqrt{\varepsilon}\sigma\int\limits_{-\infty}^{\infty}\xi\left(s\right)h\left(t-s\right)ds \tag{10}$$

Computing from (10), noting (9), one obtains

$$\begin{aligned}
&\left\langle u^2\left(t\right)\right\rangle = \varepsilon\sigma^2\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left\langle \xi\left(s_1\right)\xi\left(s_2\right)\right\rangle h\left(t-s_1\right)h\left(t-s_2\right)ds_1ds_2\\
=\ & \frac{\varepsilon\sigma^2}{2\omega_d^2}\left(\frac{1}{2\varepsilon\alpha} - \frac{1}{2\omega_0^2}\varepsilon\alpha\right) + \frac{\varepsilon\sigma^2}{2\omega_d^2}e^{-2\varepsilon\alpha t}\left\{-\frac{1}{2\varepsilon\alpha} - \frac{1}{2\omega_0^2}\left[-\varepsilon\alpha\cos\left(2\omega_d t\right) + \omega_d\sin\left(2\omega_d t\right)\right]\right\}\\
=\ & \frac{\sigma^2\left(\omega_0^2 - \varepsilon^2\alpha^2\right)}{4\omega_d^2\alpha\omega_0^2} + \frac{\varepsilon\sigma^2}{2\omega_d^2}e^{-2\varepsilon\alpha t}\left\{-\frac{1}{2\varepsilon\alpha} - \frac{1}{2\omega_0^2}\left[-\varepsilon\alpha\cos\left(2\omega_d t\right) + \omega_d\sin\left(2\omega_d t\right)\right]\right\}\\
=\ & \frac{\sigma^2}{4\alpha\omega_0^2} + \frac{\varepsilon\sigma^2}{2\omega_d^2}e^{-2\varepsilon\alpha t}\left\{-\frac{1}{2\varepsilon\alpha} - \frac{1}{2\omega_0^2}\left[-\varepsilon\alpha\cos\left(2\omega_d t\right) + \omega_d\sin\left(2\omega_d t\right)\right]\right\}
\end{aligned} \tag{11}$$

When t gets large, Eq. (11) gives the mean square of stationary response u as follows

$$\left\langle u^2 \right\rangle = \frac{\sigma^2}{4\omega^2 h}. \tag{12}$$

By squaring both sides of (2), then taking mathematical expectation and noting (3), one obtains the mean square response of Eq. (1) to be

$$\left\langle x^2 (t) \right\rangle = \left\langle u^2 (t) \right\rangle + \left\langle m^2 (t) \right\rangle . \tag{13}$$

Substituting (8) and (12) into (13), one gets the exact solution of the Eq. (1) in the form of

$$\begin{aligned} \left\langle x^2 (t) \right\rangle &= \frac{\sigma^2}{4\alpha\omega^2} + \frac{\varepsilon^2 P^2}{2\left[\left(\omega^2 - \nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2\right]} \\ &\quad + \frac{\varepsilon^2 P^2 \left[\left(\omega^2 - \nu^2\right)^2 - 4\varepsilon^2\alpha^2\nu^2\right]}{2\left[\left(\omega^2 - \nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2\right]^2} \cos(2\nu t) \\ &\quad + \frac{2\varepsilon^3\nu\alpha P^2 \left(\omega^2 - \nu^2\right)}{\left[\left(\omega^2 - \nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2\right]^2} \sin(2\nu t) . \end{aligned} \tag{14}$$

It is seen from Eq. (14) that the mean square response of Eq. (1) is time varying.

3 Numerical results

Taking time-averaging Eq. (14) over one period gives

$$\left\langle \left\langle x^2 (t) \right\rangle \right\rangle_t = \frac{\sigma^2}{4\alpha\omega^2} + \frac{\varepsilon^2 P^2}{2\left[\left(\omega^2 - \nu^2\right)^2 + 4\varepsilon^2\alpha^2\nu^2\right]} \tag{15}$$

Furthermore, Eq. (1) can be solved by stochastic averaging method in Cartesian coordinates (b, d) by the following transformation

$$\begin{aligned} x &= b\cos\nu t + d\sin\nu t, \\ \dot{x} &= -b\nu\sin\nu t + d\nu\cos\nu t, \end{aligned} \tag{16}$$

where b and d are assumed to be slowly varying random processes. Following stochastic averaging method, Eq. (1) is replaced by its averaged version [6,7] which has the corresponding Fokker-Planck equation, written for stationary probability density function $W(b, d)$, solved exactly by the technique of auxiliary function [8]. Thus, the exact stationary probability density function $W(b,d)$ of Eq. (1) takes the form (see [7] for details)

$$W(b,d) = C\exp\left\{-\frac{2h\nu^2}{\sigma^2}\left(b^2+d^2\right)+\frac{4Ph\nu^2}{\sigma^2\left(4h^2\nu^2+\Delta^2\right)}\left(\Delta b+2h\nu d\right)\right\}. \quad (17)$$

Here, $\Delta = (\omega^2-\nu^2)/\varepsilon$. By squaring both sides of the first equation in (16) and then taking mathematical expectation, one obtains

$$\left\langle x_{av}^2(t)\right\rangle = \left\langle b^2\right\rangle\cos^2\nu t + \left\langle d^2\right\rangle\sin^2\nu t + \langle bd\rangle\sin 2\nu t. \quad (18)$$

Taking averaging Eq. (18) with respect to time yields the following expression

$$\left\langle\left\langle x_{av}^2(t)\right\rangle\right\rangle_t = \frac{1}{2}\left(\left\langle b^2\right\rangle+\left\langle d^2\right\rangle\right). \quad (19)$$

Using (17), Eq. (19) is equivalent to

$$\left\langle\left\langle x_{av}^2(t)\right\rangle\right\rangle_t = \frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left(b^2+d^2\right)W(b,d)\,dbdd. \quad (20)$$

The above result is employed to check the accuracy of the present technique. In order to check the accuracy of the present technique, the various values of response of Eq. (1), denoted by $\left\langle x^2\right\rangle_{present}$, are compared to the numerical simulation results, denoted by $\left\langle x^2\right\rangle_{sim}$, and results obtained by the stochastic averaging method, denoted by $\left\langle x^2\right\rangle_{av}$, versus the particular parameter. The numerical simulation of the mean square response is obtained by 10,000-realization Monte Carlo simulation. In Table 1, time-averaging values of mean-square response of the system is performed for computation with various values of the parameter ν in the primary resonant region. The system parameters are chosen to be $\varepsilon = 0.02, \alpha = 1, \omega = 1, P = 1,\ \sigma^2 = 1$. It is seen that the stochastic averaging method gives a good prediction only when ν very closes to the natural frequency of the system. Table 2 presents time-averaging values of mean-square response of the system evaluated versus the parameter σ^2 with the system parameters chosen to be $\varepsilon = 0.02$, $\alpha = 1$, $\omega = 1$, $\nu = 1.01$, $P = 1$. Table 3 presents time-averaging values of mean-square response of the system evaluated versus the parameter P with the system parameters: $\varepsilon = 0.02, \alpha = 1$, $\omega = 1$, $\nu = 1.01$, $\sigma^2 = 1$. The tables show that the present technique gives a good prediction. The errors in the tables are defined as

$$\mathrm{Err}_{av} = \frac{\left|\left\langle x^2\right\rangle_{sim}-\left\langle x^2\right\rangle_{av}\right|}{\left\langle x^2\right\rangle_{sim}}\times 100\%, \quad (21)$$

$$\mathrm{Err}_{present} = \frac{\left|\left\langle x^2\right\rangle_{sim}-\left\langle x^2\right\rangle_{present}\right|}{\left\langle x^2\right\rangle_{sim}}\times 100\%. \quad (22)$$

Table 1. The error between the simulation result and approximate values of the averaging with respect to time of mean square response $\left\langle x^2(t)\right\rangle$ versus the parameter ν ($\varepsilon = 0.02, \alpha = 1, \omega = 1, P = 1, \sigma^2 = 1$).

ν	$\left\langle x^2\right\rangle_{sim}$	$\left\langle x^2\right\rangle_{av}$	Err_{av} (%)	$\left\langle x^2\right\rangle_{present}$	$\mathrm{Err}_{present}$ (%)
1.01	0.3458	0.3433	0.72	0.3482	0.70
1.05	0.2679	0.2431	9.28	0.2663	0.60
1.10	0.2544	0.2110	17.08	0.2543	0.03
1.15	0.2535	0.1909	24.67	0.2519	0.62
1.20	0.2500	0.1746	30.14	0.2510	0.42

Table 2. The error between the simulation result and approximate values of the averaging with respect to time of mean square response $\left\langle x^2(t)\right\rangle$ versus the parameter σ^2 ($\varepsilon = 0.02$, $\alpha = 1$, $\omega = 1$, $\nu = 1.01$, $P = 1$).

σ^2	$\left\langle x^2\right\rangle_{sim}$	$\left\langle x^2\right\rangle_{av}$	Err_{av} (%)	$\left\langle x^2\right\rangle_{present}$	$\mathrm{Err}_{present}$ (%)
0.1	0.1240	0.1227	1.058	0.1232	0.66
0.5	0.2222	0.2208	0.625	0.2232	0.48
1.0	0.3504	0.3433	2.016	0.3482	0.61
2.0	0.6017	0.5884	2.214	0.5982	0.58
5.0	1.3583	1.3236	2.555	1.3482	0.74

Table 3. The error between the simulation result and approximate values of the averaging with respect to time of mean square response $\left\langle x^2(t)\right\rangle$ versus the parameter P ($\varepsilon = 0.02$, $\alpha = 1$, $\omega = 1$, $\nu = 1.01$, $\sigma^2 = 1$).

P	$\left\langle x^2\right\rangle_{sim}$	$\left\langle x^2\right\rangle_{av}$	Err_{av} (%)	$\left\langle x^2\right\rangle_{present}$	$\mathrm{Err}_{present}$ (%)
0.1	0.2492	0.2461	1.26	0.2510	0.71
1.0	0.3486	0.3433	1.52	0.3482	0.11
2.0	0.6488	0.6380	1.67	0.6429	0.91
3.0	1.1349	1.1291	0.51	1.1340	0.08
5.0	2.7301	2.7007	1.08	2.7056	0.90

4 Summary and conclusions

In mechanical approach to a non-autonomous linear stochastic differential equation, the original system is separated into two uncoupled differential equations: ordinary differential equation and stochastic differential equation, which can be solved exactly. The mean square responses of the system obtained by the present technique are validated by numerical simulation results, obtained by Monte-Carlo simulation, and are compared to results obtained by stochastic averaging method. This technique to the authors' knowledge is not mathematically proved so far. Thus, it needs investigating and modifying further in order to make it be applicable to various kinds of stochastic differential equations.

References

[1] C.E. Laurence. *An introduction to stochastic differential equations* (version 1.2), Department of Mathematics, UC Berkeley (2002).

[2] B. Oksendal. *Stochastic Differential Equations - An introduction with Application*, Springer (2005).

[3] N.D. Anh , N.N. Hieu. *The Duffing oscillator under combined periodic and random excitations.* J. Probabilistic Engineering Mechanics (2012)30: 27-36.

[4] T. K. Caughey *Response of Van der Pol's oscillator to random excitation*, Journal of Applied Mechanics (1959) 26, 345-348.

[5] K. Piszczek. *Influence of random disturbances on determined nonlinear vibration*, in Stochastic Problems in Dynamics (B. L. Clarkson, editor) London, Pitman (1977).

[6] Y.A. Mitropolskii. *Averaging method in non-linear mechanics.* Int. J. Non-linear Mechanics. Pergamoa Press Ltd. (1967) 2: 69-96.

[7] Y.A. Mitropolskii, N.V. Dao, N.D. Anh. *Nonlinear oscillations in systems of arbitrary order.* Kiev: Naukova- Dumka (in Russian) (1992).

[8] N.D. Anh. *Two methods of integration of the Kolmogorov-Fokker-Planck equations* (English). Ukr. Math. J. (1986);38: 331-334; translation from Ukr. Mat. Zh. (1986) 38(3): 381-385.

[9] J.I. Daniel. *Engineering vibration.* New Jersy: Prentice Hall (2008).

[10] L. Lutes, S. Sarkani. *Stochastic Analysis of Structural Dynamics.* Upper Saddle River, New Jersey: Prentice Hall (1997).