

DISCRETE HEDGING WITH LIQUIDITY RISK

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Abstract

In this paper, we study European options with Black-Scholes model in a market with liquidity costs. We prove that delta hedging is still an optimal strategy. The option price in the presence of liquidity costs is given by solving a partial differential equation. Then, the applied implicit finite difference method is showed to be stable. Finally, some experiments illustrate the efficiency of our method.

1 Introduction

Liquidity risk is typically reflected in price movements: if we buy a larger number of a security, the average buy price will be higher and if we sell a larger number of a security the the average sell price will be lower. Then, size of a trade - trading volume - should be incorporated in price of a security, for example, see Jarrow (1992), Back (1993), Cvitanic and Ma (1996), Duffie and Ziegler (2003), Çetin et al. (2004), and Ku et al. (2012). Among these works, Ku et al. (2012) developed a discrete hedging with liquidity risk. They derived a partial differential equation for the option value and proposed a hedging strategy

Key words: discrete time, Black-Scholes model, liquidity cost, delta hedging.

in discrete time with the presence of liquidity costs. They also proposed the modified PDE to provide discrete time hedging strategies.

Using the same framework in Ku et al. (2012), we study how the classical hedging strategies should be modified and how the prices of derivatives should be changed in a financial market with liquidity costs, especially when we hedge only at discrete time points. We consider a discrete time version of the Black-Scholes model and a multiplicative supply curve without assuming that interest rate is zero. We prove that delta hedging is still an optimal strategy in the presence of liquidity risks. We also obtain a nonlinear partial differential equation which requires the expected hedging error converging to zero as number of hedging time goes to infinity. We provide an approximate method for solving this equation using a series solution.

The remaining of the paper is organized as follows. Section 2 introduces the model, and presents the optimal hedging strategy and a pricing PDE. Section 3 studies an analytic solution for the pricing PDE and numerical results are also provided.

2 Optimal hedging strategy

Let $S(t, 0) = S_t$ be the marginal price of the supply curve. We assume the price process S_t follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad 0 \leq t \leq T \quad (1)$$

where the drift μ is a constant, the volatility σ is a positive real number, W is a standard Brownian motion, and T is the terminal time of an European contingent claim $C(C = g(S_T)$ for some function g) of interest. In this study, we are concerned with discrete time hedging and pricing. Let us consider equally spaced times $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = T$. Set $\Delta t = t_i - t_{i-1}$ for $i = 1, \dots, n$. We consider the following discrete time version of (1)

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma Z \sqrt{\Delta t}, \quad (2)$$

where Z is a standard normal variable, and assume a multiplicative supply curve

$$S(t, x) = f(x)S(t, 0),$$

where f is a smooth and increasing function with $f(0) = 1$.

Because we consider discrete time trading, the total liquidity costs up to time T is

$$L_T = \sum_{0 \leq u \leq T} \Delta X_u [S(u, \Delta X_u) - S(u, 0)], \quad (3)$$

where $L_{0-} = 0$, and $L_0 = X_0 [S(0, X_0) - S(0, 0)]$. For the detailed structure of the liquidity costs, see Section 2 of Ku (2012). Note that the liquidity cost is always non-negative, since $S(t, x)$ is an increasing function of x . Then, the total liquidity costs up to time T could rewrite as

$$L_T = \sum_{i=1}^n \Delta X_i [S(t_i, \Delta X_i) - S(t_i, 0)] + X_0 [S(0, X_0) - S(0, 0)], \quad (4)$$

where $\Delta X_i = X_{t_i} - X_{t_{i-1}}$. X_t represents the trader's aggregate stock holding at time t (unit of money market account). Here, X_t is predictable and optional processes with $X_{0-} \equiv 0$.

We let C_0 denote the value at time 0 of contingent claim C so that the hedging error inclusive of liquidity costs is

$$H = \sum_{i=0}^{n-1} X_{t_i} (S_{t_{i+1}} - S_{t_i}) - C + C_0 - L_T - \Delta B. \quad (5)$$

where B is money market account with

$$\Delta B = rB\Delta t = r(XS - C)\Delta t.$$

Let us consider a European call option C expiring at T with strike price K , and a hedging strategy X . Let $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ be equally spaced trading times and $\Delta t = t_i - t_{i-1}$ for $i = 1, \dots, n$. A perfect hedging strategy will produce a zero hedging error with probability 1. But, considering discrete trading and liquidity costs, it is not possible to produce a strategy whose hedging error equals 0. We derive a partial differential equation which provides discrete time hedging strategies whose expected hedging error approaches zero as the length of the revision interval goes to zero.

Theorem 1. *The expected hedging error over the period $[0, T]$ approaches 0 as Δt goes to 0 imply that $X = C_{S|S=S_{t_i}}$ is an optimal hedging strategy.*

Proof. We have the hedging error over each revision interval is:

$$\Delta H = X\Delta S - \Delta B - \Delta C - \Delta X (S(t, \Delta X) - S(t, 0))$$

Recall that

$$\begin{aligned} \Delta B &= rB\Delta t = r(XS - C)\Delta t \\ \Delta S &= \mu S\Delta t + \sigma SZ\sqrt{\Delta t} \\ (\Delta S)^2 &= \sigma^2 S^2 Z^2 \Delta t + \mathcal{O}(\Delta t^{3/2}) \\ (\Delta S)^k &= \mathcal{O}(\Delta t^{3/2}), \quad k = 3, 4, 5, \dots \end{aligned}$$

Over the small time interval $[t_{i-1}, t_i]$, $i = \overline{1, n}$ we consider the change in the call option value is:

$$\begin{aligned}\Delta C &= C(S + \Delta S, t + \Delta t) - C(S, t) = C_S \Delta S + C_t \Delta t + \frac{1}{2} C_{SS} (\Delta S)^2 + \mathcal{O}(\Delta t^{3/2}) \\ &= C_S \Delta S + C_t \Delta t + \frac{1}{2} C_{SS} \sigma^2 S^2 Z^2 \Delta t + \mathcal{O}(\Delta t^{3/2})\end{aligned}$$

The liquidity cost at each interval is:

$$\Delta X (S(t, \Delta X) - S(t, 0)) = \Delta X (f(\Delta X) - 1) S(t, 0)$$

By Taylor expansion with $f(0) = 1$ we have

$$f(\Delta X) - 1 = f(\Delta X) - f(0) = f'(0) \Delta X + \frac{f''(0)}{2} (\Delta X)^2 + \mathcal{O}((\Delta X)^3)$$

The liquidity cost becomes

$$\begin{aligned}\Delta X (S(t, \Delta X) - S(t, 0)) &= \Delta X \left(f'(0) \Delta X + \frac{f''(0)}{2} (\Delta X)^2 \right) S + \mathcal{O}(\Delta t^{3/2}) \\ &= f'(0) S (\Delta X)^2 + \mathcal{O}(\Delta t^{3/2})\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta H &= X \Delta S - C_S \Delta S - C_t \Delta t - \frac{1}{2} \sigma^2 S^2 Z^2 C_{SS} \Delta t - r B \Delta t - f'(0) S (\Delta X)^2 + \mathcal{O}(\Delta t^{3/2}) \\ &= \Delta S (X - C_S) + \Delta t \left(-C_t - \frac{1}{2} \sigma^2 S^2 Z^2 C_{SS} - r (XS - C) \right) - f'(0) S (\Delta X)^2 + \mathcal{O}(\Delta t^{3/2})\end{aligned}$$

Taking expectation,

$$E(\Delta H) = \Delta S (X - C_S) + \Delta t \left(-C_t - \frac{1}{2} \sigma^2 S_t^2 C_{SS} - r (XS - C) \right) - E[(\Delta X)^2] f'(0) S + \mathcal{O}(\Delta t^{3/2})$$

Since the expected hedging error over the period $[0, T]$ approaches 0 for any S when Δt becomes small, the total hedging error over $[0, T]$ is

$$E\left(\sum \Delta H\right) = \sum E(\Delta H) \sim \mathcal{O}(\Delta t)$$

which imply

$$\Delta S (X - C_S) = 0 \tag{6}$$

and

$$\Delta t \left(-C_t - \frac{1}{2} \sigma^2 S_t^2 C_{SS} - r (XS - C) \right) - E[(\Delta X)^2] f'(0) S = 0 \tag{7}$$

From (6), we have

$$X = C_{S|S=S_{t_i}}.$$

□

Corollary 1. Let $C(S, t)$ denote the solution of the following partial differential equation

$$C_t + \frac{1}{2}\sigma^2 S_t^2 C_{SS} + r(C_S S - C) + f'(0)S_t^3 \sigma^2 C_{SS}^2 = 0 \quad \text{for all } t \in [0, T], S \geq \mathfrak{B}$$

with the terminal condition $C(S, T) = (S - K)^+$. The expected hedging error, using the hedging strategy $X = C_{S|S=S_{t_i}}$, over the period $[0, T]$ approaches 0 as Δt goes to 0.

Proof. The change in the hedging strategy is

$$\Delta X = C_S(S + \Delta S, t + \Delta t) - C_S(S, t) = C_{SS}\Delta S + C_{St}\Delta t + \frac{1}{2}C_{SSS}(\Delta S)^2 + \mathcal{O}(\Delta t^{3/2})$$

and

$$(\Delta X)^2 = C_{SS}^2(\Delta S)^2 + \mathcal{O}(\Delta t^{3/2}) = C_{SS}^2\sigma^2 S_t^2 Z^2 \Delta t + \mathcal{O}(\Delta t^{3/2})$$

Taking expectation, we have

$$E[(\Delta X)^2] = C_{SS}^2\sigma^2 S_t^2 \Delta t + \mathcal{O}(\Delta t^{3/2})$$

Therefore, (7) becomes

$$\Delta t(C_t + \frac{1}{2}\sigma^2 S_t^2 C_{SS} + r(C_S S - C) + f'(0)S_t^3 \sigma^2 C_{SS}^2) = 0$$

If C satisfies PDE following

$$C_t + \frac{1}{2}\sigma^2 S_t^2 C_{SS} + r(C_S S - C) + f'(0)S_t^3 \sigma^2 C_{SS}^2 = 0$$

with the boundary condition $C(x, T) = (x - K)^+$, then the expected hedging error over the period $[0, T]$ approaches 0 as Δt becomes small. \square

Corollary 2. The value of the discrete time delta-hedging strategy ($X = C_S, Y = C - C_S$) where C is the solution of the PDE in Theorem 1 converge almost surely to the payoff of the option $(S - K)^+$ including liquidity costs, as $\Delta t \rightarrow 0$.

Proof. Let ΔH_i denote by the hedging error over time interval $[t_{i-1}, t_i]$. We note that (after dropping the subscripts i 's of ΔH_i)

$$E[(\Delta H)^2] = E\left[\left(-C_t \Delta t - \frac{1}{2}C_{SS}(\Delta S)^2 - r(C_S S - C) - f'(0)C_{SS}^2\sigma^2 Z^2 S^3 \Delta t\right)^2\right] \leq M(\Delta t)^2,$$

for some constant M over all $t \in [0, T]$ because of the smoothness condition on C .

By the Law of Large Numbers for Martingales (see Feller, 1970, p. 243),

$$E \left[\frac{\Delta H_i}{\Delta t} \middle| \mathcal{F}_{t_{i-1}} \right] = 0,$$

for all i and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} E \left[\left(\frac{\Delta H_i}{\Delta t} \right)^2 \right] \leq M \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

imply that

$$\frac{1}{n} \sum_{i=1}^n \frac{\Delta H_i}{\Delta t} = \frac{\Delta t}{T} \sum_{i=1}^n \frac{\Delta H_i}{\Delta t} = \frac{1}{T} \sum \Delta H_i \rightarrow 0$$

almost surely as $\Delta t \rightarrow 0$. This leads to almost sure convergence of the total hedging error $\sum \Delta H_i$ as Δt tend to 0.

□ **Remark.** The expected hedging error over each revision interval is expressed similarly to the one studied by Ku et al. (2012). However, our approach to the problem differs from that of Ku et al. because we don't use assumption of zero interest rate. Moreover, we proved that the delta hedging is still an optimal strategy in the presence of liquidity risks.

3 Analytic solution of PDE

In this section, we discuss a possible analytic solution for the pricing equation derived in Section 2. We explain briefly how to approximate the option value including liquidity costs using a series solution.

Let $\alpha = f'(0)$. Now (8) is written as

$$C_t + \frac{1}{2} \sigma^2 S^2 C_{SS} + \alpha \sigma^2 S^3 C_{SSS} + r(C_S S - C) = 0 \quad \text{for all } t \in [0, T], S \geq \mathbb{Q}$$

with the boundary condition $C(S, T) = (S - K)^+$.

For sufficiently small $\alpha > 0$, we seek a solution in the form of

$$C(x, t) = C_0(x, t) + \alpha C_1(x, t) + \alpha^2 C_2(x, t) + \dots = C_0(x, t) + \alpha C_1(x, t) + \mathcal{O}(\alpha^2), \quad (10)$$

Inserting (10) into (9), we obtain the following equations for $C_0(S, t)$ and $C_1(S, t)$:

$$\frac{\partial C_0}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_0}{\partial S^2} + rS \frac{\partial C_0}{\partial S} - rC_0 = 0 \quad \text{for all } t \in [0, T], S \geq 0, \quad (11)$$

with the condition $C_0(S, T) = (S - K)^+$, and

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + \sigma^2 S^3 \left(\frac{\partial^2 C_0}{\partial S^2} \right)^2 + rS \frac{\partial C_1}{\partial S} - rC_1 = 0 \quad (12)$$

for all $t \in [0, T]$, $S \geq 0$, with the condition $C_1(S, T) = 0$.

The solution to the Black-Scholes partial differential (11) is well-known as

$$C_0(S, t) = SN(d+) - Ke^{-r(T-t)}N(d-),$$

where

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{z^2}{2}} dz,$$

$$d+ = \frac{1}{\sigma\sqrt{T-t}} \left[\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2} \right) (T-t) \right],$$

$$d- = \frac{1}{\sigma\sqrt{T-t}} \left[\ln \frac{S}{K} + \left(r - \frac{\sigma^2}{2} \right) (T-t) \right],$$

Also, it is known that

$$\frac{\partial^2 C_0}{\partial S^2} = \frac{1}{\sigma S \sqrt{T-t}} N'(d+) = \frac{1}{\sigma S \sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d+)^2}. \quad (13)$$

Substituting(13) into (12), we obtain

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + rS \frac{\partial C_1}{\partial S} - rC_1 + \frac{S}{2\pi(T-t)} e^{-\frac{[\ln \frac{S}{K} + (\frac{r+\frac{\sigma^2}{2}}{\sigma^2})(T-t)]^2}{\sigma^2(T-t)}} = 0 \quad (14)$$

with the condition $C_1(S, T) = 0$.

The result of this PDE can be solved by using Implicit Finite Difference Method. The time-space grid of points is created over which the approximate solution is computed.

Time domain $t \in [0, T]$ is discretized by $(N + 1)$ points evenly spaced with time step h :

$$0 = t_0 < t_1 = \delta t < \dots < t_N = N\delta t = T$$

The natural domain is $(-\infty, +\infty)$ is truncated to $[0, S_{max}]$, and discretized by an $(M + 1)$ point uniform grid with spacial step δS ,

$$0 = S_0 < S_1 = \delta S < \dots < S_M = M\delta S = S_{max}$$

We build a mesh consisting $(N + 1) \times (M + 1)$ points (t_i, x_j) , and denote the solution of the PDE at these mesh points

$$f_{i,j} = C_1(t_i, S_j) \quad \text{for } 0 \leq i \leq N, 0 \leq j \leq M$$

For the implicit method, (14) is discretized using the following formulas

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{f_{i+1,j} - f_{i,j}}{\delta t} \\ \frac{\partial f}{\partial S} &= \frac{f_{i,j+1} - f_{i,j-1}}{2\delta S} \\ \frac{\partial^2 f}{\partial S^2} &= \frac{f_{i,j+1} - f_{i,j-1} - 2f_{i,j}}{(\delta S)^2}\end{aligned}$$

where the indices i and j represent nodes on the pricing grid.

Substituting these approximations into the PDE gives,

$$\frac{f_{i+1,j} - f_{i,j}}{\delta t} + rj\delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\delta S} + \frac{1}{2}\sigma^2(j\delta S)^2 \frac{f_{i,j+1} - f_{i,j-1} - 2f_{i,j}}{(\delta S)^2} - rf_{i,j} + W_{i,j} = 0$$

where $W_{i,j} = \frac{j\delta S}{2\pi(T-i\delta t)} e^{-\frac{[\ln \frac{i\delta S}{K} + (r+\frac{\sigma^2}{2})(T-i\delta t)]^2}{\sigma^2(T-i\delta t)}}$,
which reduces to

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} + W_{i,j} \quad (15)$$

where

$$\begin{aligned}a_j &= \frac{1}{2}\delta t(rj - \sigma^2 j^2) \\ b_j &= 1 + \delta t(\sigma^2 j^2 + r) \\ c_j &= \frac{1}{2}\delta t(-rj - \sigma^2 j^2)\end{aligned}$$

Since the equations are solved working backwards in time, superficially (15) says that three unknowns must be calculated from only one known value. This is shown pictorially in the following diagram,

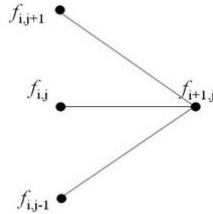


Figure 1: Implicit Finite Difference Viewed as a Pseudo-Trinomial Tree

However, when all the equations at a given time point are written simultaneously there are $M - 1$ equations in $M - 1$ unknowns. Hence the value for f at each node can be calculated uniquely.

In the option pricing framework, given the option payoff at expiry nodes then the prices δt before expiry can be calculated, then from those prices the value $2\delta t$ before expiry can be calculated, and working iteratively backwards through time until the option price at grid nodes for $t = 0$ (i.e. today) can be calculated.

The formulation for the implicit method given in equation

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} + W_{i,j}$$

could be written in the matrix notation

$$BF_i = F_{i+1} + K_i + W_i \quad i = N - 1, \dots, 1, 0$$

where

$$\begin{aligned} F_i &= [f_{i,1} \quad f_{i,2} \quad \dots \quad f_{i,M-1}]^T \\ K_i &= [-a_1 f_{i,0} \quad 0 \quad \dots \quad 0 \quad -c_{M-1} f_{i,M}]^T \end{aligned}$$

and

$$B = \begin{bmatrix} b_1 & c_1 & 0 & \dots & 0 & 0 \\ a_2 & b_2 & c_2 & \dots & 0 & 0 \\ 0 & a_3 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{M-1} & b_{M-1} \end{bmatrix}$$

Theorem 2. *The infinity norm of B^{-1} is always less than 1 then the Implicit Finite Difference Methods for (14) converges, or is stable for all values of $\rho, \sigma, \delta t$.*

Proof. The formulation for the implicit method given in equation

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} + W_{i,j}$$

is written in the matrix notation

$$BF_i = F_{i+1} + K_i + W_i \quad i = N - 1, \dots, 1, 0 \quad (16)$$

An iterative algorithm that is unstable will lead to the calculation of ever increasing numbers that will at some point approach infinity. On the other hand, a stable algorithm will converge to a finite solution. Typically the faster that finite solution is reached the better the algorithm.

From standard results in matrix algebra it is known that a matrix equation of the form given in (16) is stable if and only if

$$\|B^{-1}\|_{\infty} \leq 1$$

Heuristically, if the infinity norm of B^{-1} is less than 1 then successive values of F_i get smaller and smaller, and hence the algorithm converges, or is stable. (Alternatively if the infinity norm of B^{-1} is greater than 1 then successive values of F_i get larger and larger and hence diverge.)

It can be shown that the infinity norm of B^{-1} is less than 1 for all values of $\rho, \sigma, \delta t$, see Section 6 of Duffy (2006). Hence the Implicit Finite Difference Method is always stable. \square We next present some numerical simulations to examine the effect of liquidity costs on option prices.

Table 1: *Option prices with liquidity costs.*

Initial spot	Liquidity cost $f'(0)$				
	0 (Black-Scholes)	0.0001	0.0005	0.001	0.002
80	1.5617	1.5711	1.6089	1.6562	1.7507
85	2.7561	2.7698	2.8246	2.8932	3.0303
90	4.4479	4.4652	4.5343	4.6207	4.7935
95	6.6696	6.6887	6.7650	6.8604	7.0513
100	9.4134	9.4320	9.5063	9.5992	9.7849
105	12.6388	12.6546	12.7181	12.7975	12.9562
110	16.2837	16.2953	16.3414	16.3990	16.5143
115	20.2769	20.2832	20.3080	20.3391	20.4013

Table 1 presents the option prices inclusive of liquidity costs with varying $f'(0)$ and initial stock price S_0 . The options prices are obtained by solving the PDE given in Theorem 1 numerically. The parameter values that we used are strike price $K = 100$, $r = 0.2$ and $T = 1$ year. Consistent with intuition, we observe that the option prices increase slightly when the parameter of liquidity costs $f'(0)$ (the slope at 0 of supply curve) increases.

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