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A TWO-DEMENSIONAL MODEL FOR CHAOTIC BEHAVIOR STUDY OF ATMOSPHERIC MERIDITIONAL CIRCULATION OF SOUTHEAST ASIA

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Abstract

The purpose of this paper is to provide a chaotic analysis of atmospheric meridional circulation (AMC) of Southeast Asia by using a two-dimensional model. The atmospheric meridional circulation is influenced by winter monsoon. This model is simplified from the atmospheric convective model with two-dimensional Cartesian coordinates (i.e. $y-z$-plane) based on Lorenz model. The bsifurcation diagram of the two-dimensional model is used for studying chaotic behavior in some range of the parameter F which represents an external forcing term (i.e. soil moisture) and parameter r which represents dimensionless convective term.

1. Introduction

Atmospheric meridional circulation (AMC) over Southeast Asia is influenced by winter monsoon. AMC can be analyzed from winter monsoon to study its chaotic behavior.

The atmospheric circulation over Southeast Asia is directly influenced by monsoon. It consists of two types which are atmospheric zonal circulation (AZC) influenced by summer monsoon and atmospheric meridional circulation (AMC) influenced by winter monsoon. This research focuses on atmospheric

Key words: Atmospheric Meridional Circulation, Winter monsoon, Lorenz model, Bifurcation diagram, Chaotic Behavior.

meridional circulation. In winter, the air over the land becomes much colder and drier than the air over the ocean. An external forcing term for northeast monsoon is soil moisture. The governing convective model (GCM) [1] can be applied with AMC but GCM has much more variables that is very complicated to analyze. Thus, GCM should be simplified to two-dimensional model in meridional-vertical plane ($y-z$- plane) because the component of velocity vector along latitude (y-axis), v, is stronger than the component of velocity vector along longitude (x-axis), u. The step of deriving the twodimensional model is shown in Section 3 as well as study of chaotic behavior with external forcing factor F (i.e. soil moisture) and dimensionless convective term r. Bifurcation analysis is the tool that is used to investigate chaotic behavior of the two-dimensional model.

[2] apply a forced two-dimensional model with an external forcing term which is represented by sea surface temperature to predict summer monsoon over Indian and Pacific oceans. The result shows that bifurcation analysis of the modified forced two-dimensional model presents the influence of slowly varying forcing on the Indian summer monsoon. [3] focus on the heavy rainfall over southeast India during the winter monsoon period in December 2001. Anticyclonic circulation in the upper troposphere over the east contributes to the bifurcation of the strong westerlies. Northern Hemisphere (the westerlies moved equatorward) and a ridge in the Southern Hemisphere. They mainly consider the speed convergence and the extension of the upper-level trough. The speed convergence resultes in moisture and mass which is converged to maximum, supporting to rising motions and rainfall. [4] study atmospheric circulation model using a modified Lorenz-84 model by neglecting the effect of the westerlies and the atmospheric eddies. They use the modified Lorenz-84 model with a range of values of parameters including two different periodic perturbative coefficients. The results show chaotic behavior by a small change in the parameters.

2. THEORY AND RELATED WORKS

This section describes the theory and related works which consist of characteristic equation, stability of equilibrium points, Lorenz model and Routh-Hurwitz stability criteria.

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2.1. Characteristic Equation

Characteristic polynomial of a matrix A is

$$|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{1}$$

where λ is the eigenvalue and I is the identity matrix.

The order of a characteristic polynomial of an $n \times n$-matrix is n and the coefficient of the term that has the highest order is 1. The characteristic equation can be written as

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{2}$$

2.2 Stability of Equilibrium Points

Let $x = 0$ be an equilibrium point of a nonlinear system $\dot{x} = f(x)$ [5], where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is the neighborhood of the origin. The Jacobian matrix A at $x = 0$ is $A = \partial f/\partial x|_{x=0}$. Let $\lambda_i, i = 1 \cdots, n$ be the eigenvalues of A. Then

1-The origin is asymptotically stable if $\Re(\lambda_i) < 0$ for all eigenvalues of A.

2-The origin is unstable if $\Re(\lambda_i) > 0$ for any eigenvalues of A.

2.3 Lorenz Model

In 1963, Edward Norton Lorenz is interested in the predictability of the solution to hydrodynamics equations. He introduces the first model that shows the chaotic behavior in numerical solution of the fluid convection of the atmosphere. The Lorenz equations are derived from Navier-Stoke equations to simple nonlinear ordinary differential equations based on two-dimensional representation of the earth's atmosphere [6]. The Lorenz equations are [7]

$$\begin{aligned} \dot{X} &= \sigma(Y - X), \\ \dot{Y} &= rX - Y - XZ, \\ \dot{Z} &= XY - bZ. \end{aligned} \tag{3}$$

The parameters are σ, b and r. The parameter σ is called the Prandtl number which represents the ratio between the fluid viscosity to its thermal conductivity. The parameter b is the ratio of the width to the height of the plane. The parameter r is called the Rayleigh number which represents the difference in temperature between the upper and lower surfaces of the fluid. The super-dot is a derivative of variable with respect to dimensionless time. All of the three parameters are given positive values. Lorenz finds that the values of parameters $\sigma = 10, b = 8/3$ and initial condition $(x_0, y_0, z_0) - (0.1.0)$ give the best representation of Equation for the earth's atmosphere.

The variables X, Y and Z are not referred to coordinates in space. For these equations, the variable x is proportional to the intensity of the convection motion which oscillations around each equilibrium point $(x_e), x_e^+$ and x_e^-. The variable Y is proportional to the temperature difference between the ascending and descending currents where similar signs of X and Y denoted that warm fluid is rising and cold fluid is descending. The variable Z is proportional to the distortion of the vertical temperature profile from linearity.

2.4 Routh-Hurwitz Stability Criterion

Let $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a polynomial, where a_i are the real constant coefficients and $i = 1, 2 \cdots, n$. Define the Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1 = [a_1], H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \text{and}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots 0 \\ a_3 & a_2 & a_1 & 1 & \cdots 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots \vdots \\ 0 & 0 & 0 & 0 & \cdots a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $p(\lambda)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive:

$$\det H_j > 0, j = 1, 2, \cdots, n. \tag{4}$$

When $n = 2$, the Routh-Hurwitz method simplify to $\det H_1 = a_1 > 0$ and

$$\det H_2 = \det \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} = a_1a_2 > 0$$

or $a_1 > 0$ and $a_2 > 0$. For polynomial of degree $n = 2, 3, 4$ and 5, the Routh-Hurwitz method can be summarized as follows

$n = 2 : a_1 > 0$ and $a_2 > 0$.

$n = 3 : a_1 > 0, a_3 > 0$ and $a_1a_2 > a_3$.

$n = 4 : a_1 > 0, a_3 > 0, a_4 > 0$ and $a_1a_2a_3 > a_3^2 + a_1^2a_4$.

$n = 5 : a_i > 0, i = 1, 2, 3, 4, 5, a_1a_2a_3 > a_3^2 + a_1^2a_4$ and
$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_5^2$.

3. COMPUTATIONAL DETAILS

3.1 Two-dimensional Model of Atmospheric Meridional Circulation

The two-dimensional model of AMC is simplified from the governing convection model to develop two-dimensional form in meridional-vertical plane ($y-z$ plane) because the component of velocity vector along latitude (y-axis), v, is stronger than the component of velocity vector along longitude (x-axis), u. The atmospheric convective model consists of the momentum equations, the thermodynamic equation and the continuity equation.

The two-dimensional model assumes incompressible fluid motion contained in a cell which has a constant higher temperature at the bottom and a lower temperature at the top while the upper and lower boundaries temperature are constants.

The two-dimensional model is a mathematical model which describes atmospheric meridional circulation that is reduced to nonlinear ordinary differential equations system by the steps based on Lorenz model.

Steps of deriving the two-dimensional model:

1) Simplify the model in $y-z$ plane: $u=0, \partial/\partial x=0, v=-\partial\psi/\partial z$ and $w=\partial\psi/\partial y$.

2) Substitute $T(y,z,t)=\overline{T}(t)+T_1(y,z,t)$ into momentum and thermodynamics equations where $\overline{T}(t)$ is the average value over the entire convection region, $T_1(y,z,t)$ is the perturbation from the average value and $\theta=\overline{T_1"}(z,t)+T'(y,z,t)$ is departure of temperature.

3) Define Fourier approximation of stream function and departure of temperature

$$\psi=\sum_{m,n}^{\infty}\psi_{min}(t)cos\frac{ma\pi y}{H}sin\frac{n\pi z}{H} \tag{5}$$

$$\theta=\sum_{m,n}^{\infty}\theta_{min}(t)sin\frac{ma\pi y}{H}sin\frac{n\pi z}{H} \tag{6}$$

where m,n are the wave number in the y and z direction. This research just focuses on ψ_{11},ψ_{02} and θ_{11}.

4) Substitute Fourier approximation in the model and arrange with dimensionless time $\tau=(1+a^2)\pi^2\kappa H^{-2}t$ which finally yields the two-dimensional model which is a nonlinear ordinary differential equations system

$$\begin{aligned}\dot{X} &= \dot{Z}-\sigma X-YZ,\\ \dot{Y} &= -b\sigma Y+XZ\\ \dot{Z} &= XY-rX-Z\end{aligned} \tag{7}$$

where X is the speed of the convective motion, Y is the speed of the convective motion in vertical direction, Z is the horizontal temperature variation, σ is Prandtl number, b is the horizontal wave vector of the convective motion and r is a proportion of Rayleigh number and critical Rayleigh number. The superdot describes derivative of variable with respect to dimensionless time. The chaos occurs in the two-dimensional model when values of parameter are $\sigma = 0.16, b = 8/3$ and $r = 2.89$.

3.2 Modified Forced Two-Dimensional Model

The AMC has many important factors such as sea surface temperature, soil moisture or some external force etc. This research also assumes that the positive $X - Z$ in the two-dimensional model represents the active phase and the negative $X - Z$ represents to the weak phase of AMC. The forcing term corresponds to soil moisture.

The two-dimensional model with forcing terms correspond to soil moisture is given as

$$\begin{aligned} \dot{X} &= -Z - \sigma X - YZ + F_x, \\ \dot{Y} &= -b\sigma Y + XZ + F_y, \\ \dot{Z} &= XY - rX - Z + F_z. \end{aligned} \tag{8}$$

where F_x, F_y and F_z , and are the forcing terms associated with X, Y and Z, respectively.

The modified forced two-dimensional model is mathematically analyzed for chaotic behavior study. This research focuses in case of $F_x = 0, F_y = \sigma brF$ and $F_z = 0$.

The modified forced two-dimensional model yields

$$\begin{aligned} \dot{X} &= -Z - \sigma X - YZ, \\ \dot{Y} &= -b\sigma Y + XZ + \sigma brF, \\ \dot{Z} &= XY - rX - Z. \end{aligned} \tag{9}$$

where $\sigma = 0.16, b = 8/3$ and $r_c = 2.89$ is a critical value of r.

3.3 Chaotic Behavior Study of the Modified Forced Two-dimensional Model

Steps of chaotic behavior study of the modified forced two-dimensional model:

1) Finding the equilibrium points, this model has three equilibrium points $x_{e_1} = \mathcal{O}(0, Fr, 0)$ and $x_{e_1}, x_{e_2} = P_{\pm}(x_{p_\pm}, y_p, z_{p_\pm})$ where

$x_{p_\pm} = \pm\left[\frac{(\Delta_F)^{3/2}}{\sigma^2 b(1+rF)} - \frac{(r-\sigma+rF+r^2F+1)\sqrt{\Delta_F}}{\sigma(rF+1)}\right]$,

$y_p = \frac{r^2F+r-\sigma}{rF+1} - \frac{\Delta_F}{\sigma b(1+rF)}$.

$z_{p_\pm} = \pm\sqrt{\Delta_F}$
$\Delta_F = \frac{\sigma b\sqrt{r^2+2r-4\sigma+1}}{2} - \sigma^2 b + \frac{\sigma b r^2 F + \sigma b r F + \sigma b r + \sigma b}{2} \frac{\sigma b r F\sqrt{r^2+2r-4}}{2}$
2) The characteristic equation is

$$\lambda^3 + \sigma b + \sigma + 1)\lambda^2 + (-x_0^2 + y_0^2 + z_0^2 - r y_0 + y_0 + \sigma - r + \sigma b + \sigma^2 b)\lambda$$
$$+ x_0 z_0 - r x_0 z_0 + 2 x_0 y_0 z_0 - \sigma x_0^2 + z_0^2 + \sigma b y_0^2 + \sigma b y_0 - \sigma b r y_0 + \sigma^2 b - \sigma b r = 0 \quad (10)$$

4. RESULTS AND DISCUSSION

The results show that the behavior of the modified forced two-dimensional model with forcing terms has a chaotic behavior in some of its parameter ranges as analyzed by bifurcation structure in the $r - F$ plane, as shown in Figure 1.

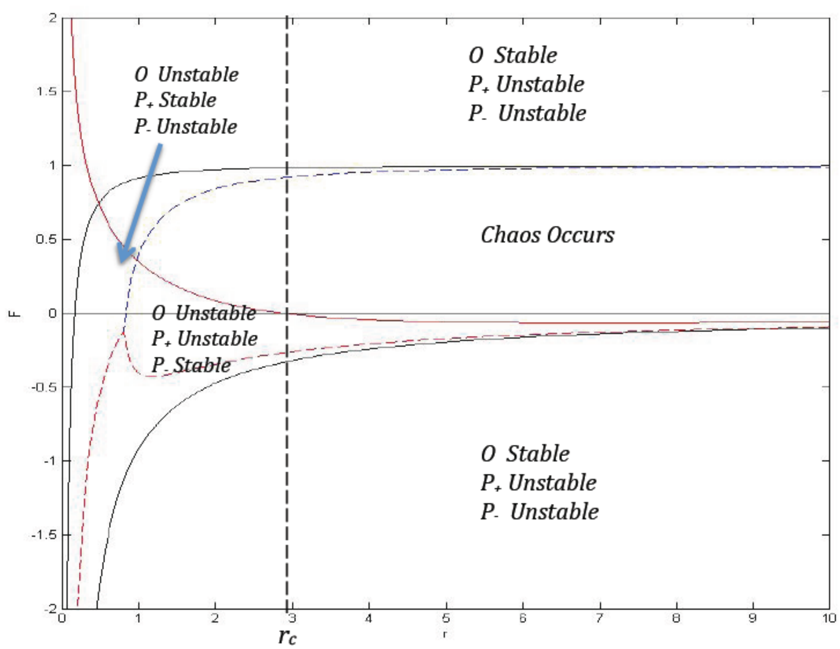


Figure 1. Bifurcation structure of the modified forced two-dimensional model in case of $F_x = 0$, $F_y = \sigma b r F$ and $F_z = 0$.

The numerical solutions of the modified forced two-dimensional model with $F = 0.2$ and $r = 0.5$ result in stability at $P+$ for 1,000 iterations as shown in Figure 2 and 3,000 iterations in Figure 3. (The equilibrium points O, P_+ and P_- are represented by green, blue and red points respectively.)

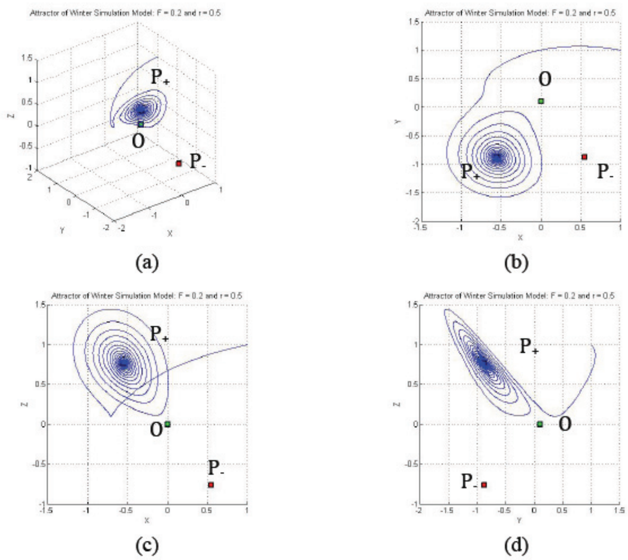


Figure 2. The solutions of the modified forced two-dimensional model of AMC with $F = 0.2$ and $r = 0.5$ (a) 3-dimensional perspectives, (b) *X*-*Y* plane, (c) *X*-*Z* plane and (d) *Y*-*Z* plane.

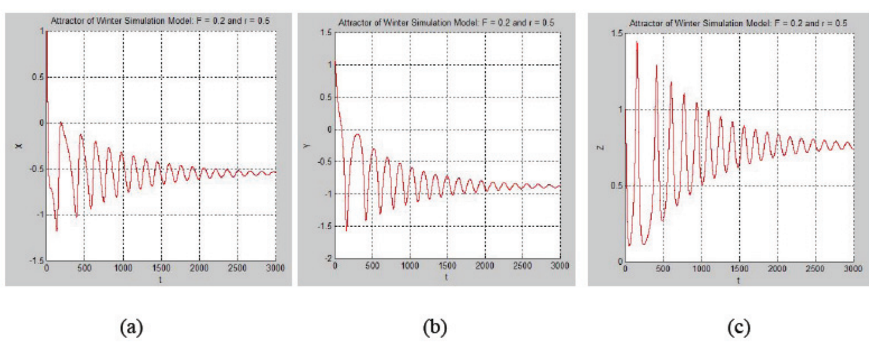


Figure 3. The graphs between *X*, *Y* or *Z* and time *t* for the modified forced two-dimensional model of AMC with $F = 0.2$ and $r = 0.5$ (a) *X*-*t* plane, (b) *Y*-*t* plane and
(c) *Z*-*t* plane.

According to [3], the rainfall oscillates non-periodically between two phases, the weak/break phase (little rainfall) and the active phase (good rainfall). Similarly, this paper assumes that the positive $X - Z$ (i.e. equilibrium point P_+) in the attractor corresponds to the active phase (strong AMC) and the negative $X - Z$ (i.e. equilibrium point P_-) corresponds to the weak phase (weak AMC).

This paper is interested in the amplitude of the convective motion X and the horizontal temperature variation Z because they directly influence AMC.

From Figures 2 and 3, the solutions of the modified forced two-dimensional model result in stability at P_+, which indicates the active phase of AMC.

The numerical solutions of the forced two-dimensional model with $F = 0.1$ and $r = 1$ result in stability at P_- for 1,000 iterations as shown in Figure 4 and 3,000 iterations in Figure 5.

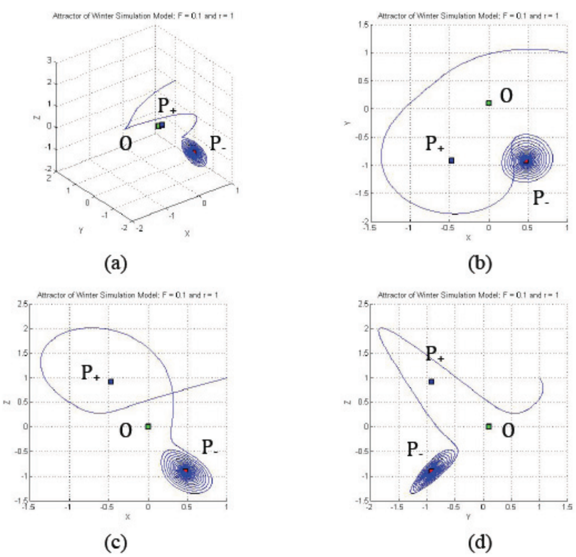


Figure 4. **The solutions of the modified forced two-dimensional model with F = 0.1 and r = 1 (a) 3-dim ensional perspectives, (b) *X*-*Y* plane, (c) *X*-*Z* plane and (d) *Y*-*Z* plane.**

From Figures 4 and 5, the solutions of forced two-dimensional model result in stability at P_-, which indicates the weak phase of AMC.

The numerical solutions of the forced two-dimensional model with $F = 0.05$ and $r = 40$ result in chaos occurs at P_- and P_+ for 1,000 iterations as shown in Figure 6 and 3,000 iterations in Figure 7.

From Figure 6, numerical solutions of the modified forced two-dimensional model with $F = 0.4$ and $r = 2$ result in chaotic behavior. Figure 7 shows the time frequencies of variables X, Y and Z that have nonperiodical patterns and oscillate between equilibrium points P_+ and P_-. That is, the AMC can not be predicted when r and F is over chaos region as shown in Figure 1.

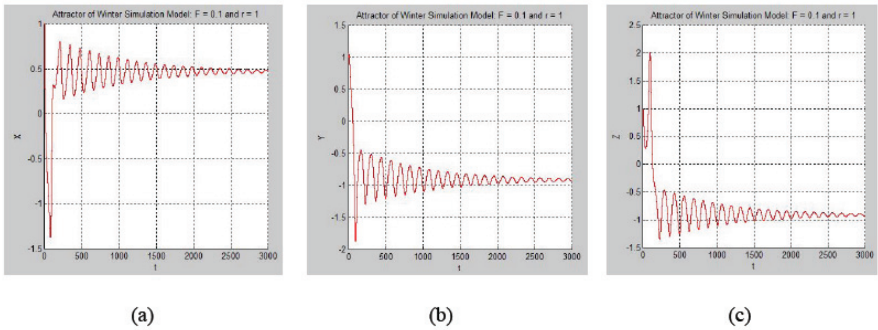


Figure 5. The graphs between X, Y or Z and time t for the modified forced two-dimensional model with $F = 0.1$ and $r = 1$ (a) X-t plane, (b) Y-t plane and (c) Z-t plane.

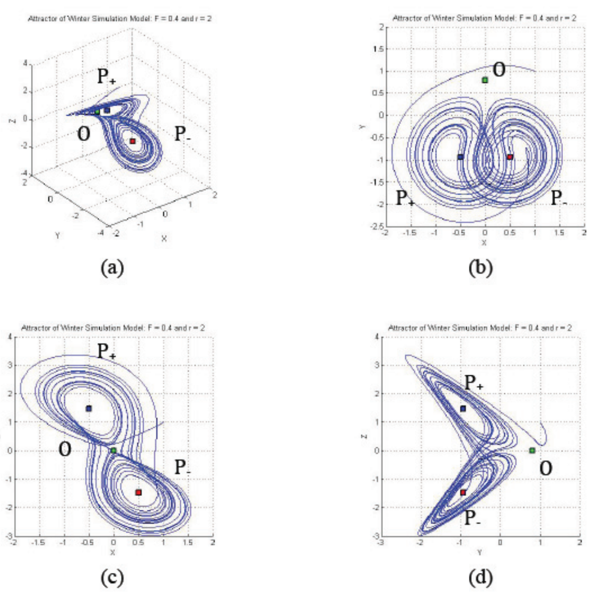
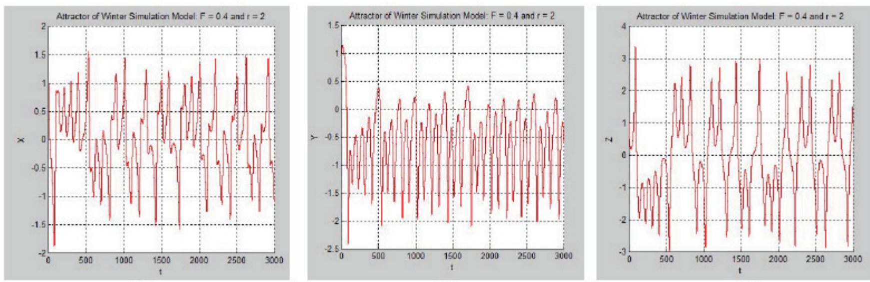


Figure 6. The solutions of the modified forced two-dimensional model with $F = 0.4$ and $r = 2$ (a) 3-dimensional perspectives, (b) X-Y plane, (c) X-Z plane and (d) Y-Z plane.

5. CONCLUSION

A modified forced two-dimensional model is used for analysis of chaotic behavior of atmospheric meridional circulation (AMC) over Southeast Asia. A



(a) (b) (c)

Figure 7. The graphs between X, Y or Z and time t for the modified forced two-dimensional model with $F = 0.4$ and $r = 2$ (a) X-t plane, (b) Y-t plane and (c) Z-t plane.

two-dimensional model is simplified form governing convective model (GCM) in meridional-vertical or $y - z$ plane. This is because in the AMC the air over the land becomes much colder than the air over the ocean so that the component of velocity vector along latitude v is stronger than the component of velocity vector along longitude u. This paper focuses on the two-dimensional model with forcing factor (i.e. soil moisture) in case of $F_x = 0, F_y = \sigma brF$ and $F_2 = 0$ (fixed parameters σ and b are 0.16 and 8/3, respectively). There are three equilibrium points for the modified forced two-dimensional model. The results show that the external forcing term F and dimensionless convective term r influence the bifurcation of this model and the modified forced two-dimensional model has chaotic behavior of P_- and P_+ in some ranges of parameters F and r.

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