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HAMMING DISTANCES OF REPEATED-ROOT CONSTACYCLIC CODES OF PRIME POWER LENGTHS OVER $\mathbb{F}_{5^m} + u\mathbb{F}_{5^m}$

Nguyen Trong Bac

Department of Basic Sciences
University of Economics and Business Administration
Thai Nguyen University, Thai Nguyen 250000, Vietnam
e-mail: bacnt2008@gmail.com

Abstract

The ring $\mathcal{R} = \mathbb{F}_{5^m} + u\mathbb{F}_{5^m}$ has precisely $5^m(5^m - 1)$ units, which are of the forms $\alpha + u\beta$ and γ, where α, β, γ are nonzero elements of the field $\mathbb{F}_{5^m}$. Using the algebraic structure in term of generator polynomials of these codes, the Hamming and symbol-pair distance distributions of all such codes are completely determined.

1. Introduction

Classically, the algebraic structures of constacyclic codes are determined by ideals of the polynomial rings over finite fields, Galois rings and finite chain rings. Recently, codes over finite non-chain rings have also been studied. In 2010, Zhu et.al. investigated the structures and properties of cyclic codes over the ring $\mathbb{F}_2 + v\mathbb{F}_2$ where $v^2 = v$. The structure of codes over the ring $\frac{\mathbb{Z}_3[v]}{\langle v^3 - v\rangle}$ is studied by Bayram and Siap. After that, Gao and Wang introduced a new generalization by considering the linear codes over $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$. In 2014, Bayram and Siap continued to study codes over the ring $\frac{\mathbb{Z}_p[v]}{\langle v^p - v\rangle}$. The algebraic structures of linear, cyclic and constacyclic codes over this ring are determined

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by means of a Gray map. By using the Gray map obtained some interesting classes of quantum error correcting codes over $\mathbb{F}_p$. Moreover, the algebraic structures of the cyclic codes of arbitrary length over the finite non-chain ring $\mathbb{F}_p + v\mathbb{F}_p + \cdots + v^{p-1}\mathbb{F}_p$ where $v^p = v$ are also introduced. As noted, this ring $\mathbb{F}_p + v\mathbb{F}_p + \cdots + v^{p-1}\mathbb{F}_p$ is in fact a direct sum of p copies of the finite field $\mathbb{F}_p$. In recent paper, the algebraic structure of λ-constacyclic codes over such finite semi-simple rings is investigated. Among others, necessary and sufficient conditions for the existence of self-dual, LCD, and Hermitian dual-containing λ-constacyclic codes over finite semi-simple rings were provided.

It is well-known from [5, 6] that the ambient ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\lambda\rangle}$ is a chain ring. Therefore, the ideals of the ring are completely determined, i.e., λ-constacyclic codes are also given. From this, in [5, 6], the Hamming distances of all such codes are obtained.

Let $x = (x_0, x_1, \ldots, x_{n-1})$ be a vector in Ξ^n, where Ξ is a code alphabet. The symbol-pair distance is given in [2] by using Hamming distance over the alphabet (Ξ, Ξ) as follows:

$$\mathrm{d_{sp}}(\mathbf{x}, \mathbf{y}) = |\{i : (x_i, x_{i+1}) \neq (y_i, y_{i+1})\}|,$$

where x, y are vectors in Ξ. Then $\mathrm{d_{sp}}(C) = \min_{\mathbf{x},\mathbf{y}\in C,\mathbf{x}\neq\mathbf{y}}\{\mathrm{d_{sp}}(\mathbf{x}, \mathbf{y})\}$ is the symbol-pair distance of code C. In addition, the symbol-pair distance of a linear code C is the minimum symbol-pair weight of nonzero codewords of the code C:

$$\mathrm{d_{sp}}(C) = \min\{\mathrm{wt_{sp}}(\mathbf{x}) \mid \mathbf{x} \neq \mathbf{0},\ \mathbf{x} \in C\}.$$

The problem of determining the symbol-pair distances is very difficult in general. Recently, we succesfully established the symbol-pair distances of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}$. Although algebraic structure of all constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are provided by Dinh in [6], Hamming and symbol-pair distances ofλ-constacyclic codes have remained open, where $\lambda \in \mathbb{F}_{p^m}$. Motivated by that, we solved this problem in this paper. Our technique can also be extended to establish the symbol-pair distance of $\alpha + u\beta$- and λ-codes, where $0 \neq \alpha, \beta, \lambda \in \mathbb{F}_{p^m}$.

2. Preliminaries

Let R be a finite commutative ring. In [7], we knew that a finite commutative ring R is a chain ring if and only if R is a local principal ideal ring. It is also equivalent to the condition that R is a local ring and the maximal ideal M of R is principal. A nonempty subset C of R^n is a code of length n over R. The code C is said to be "linear" if C is an R-submodule of R^n.

For an invertible element λ of R, the λ-constacyclic (λ-twisted) shift τ_λ on R^n is the shift

$$\tau_\lambda(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \cdots, x_{n-2}),$$

and a code C is said to be λ-constacyclic if $\tau_\lambda(C) = C$.

Each codeword $c = (c_0, c_1, \ldots, c_{n-1})$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$, and the code C is in turn identified with the set of all polynomial representations of its codewords. Then in the ring $\frac{R[x]}{\langle x^n - \lambda\rangle}$, $xc(x)$ corresponds to a λ-constacyclic shift of $c(x)$. From that, the following fact is well-known and straightforward:

Proposition 2.1. *A linear code C of length n is λ-constacyclic over R if and only if C is an ideal of $\frac{R[x]}{\langle x^n - \lambda\rangle}$.*

For any nonzero $\lambda \in \mathbb{F}_{p^m}$, linear λ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ are precisely the ideals of the ambient ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$.

By applying the Division Algorithm, there are nonnegative integers k_q, k_r such that $s = k_q m + k_r$, and $0 \le k_r \le m-1$. Let $\lambda_0 = \lambda^{p^{(k_q+1)m-s}} = \lambda^{p^{m-k_r}}$. Then $\lambda_0^{p^s} = \lambda^{p^{(k_q+1)m}} = \lambda$. As discussed in [6, Section 3], it is simple to verify that in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$, $(x - \lambda_0)^{p^s} = x^{p^s} - \lambda_0^{p^s} = x^{p^s} - \lambda = 0$. Therefore, $x - \lambda_0$ is a nilpotent element of $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$ with the nilpotency index p^s. This implies that $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$ has a maximal ideal which is $\langle x - \lambda_0\rangle$ and hence, $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$ is a chain ring. Then the structure of all λ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ and their duals are determined as follows.

Theorem 2.2. *λ-constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ are precisely the ideals $\langle (x-\lambda_0)^i\rangle$, $i = 0, 1, \ldots, p^s$, of the chain ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$. Each λ-constacyclic code $C_i = \langle (x-\lambda_0)^i\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda\rangle}$ has $p^{m(p^s - i)}$ codewords. The dual of C_i is the λ^{-1}-constacyclic code $C_i^\perp = \langle (x - \lambda_0^{-1})^{p^s - i}\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \lambda^{-1}\rangle}$, which contains p^{mi} codewords.*

The number of nonzero components of a codeword $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{F}_{p^m}^n$ is defined as the *Hamming weight* of $\mathbf{x}$, denoted by $\mathrm{wt_H}(\mathbf{x})$. The number of components in two codewords x, y which they differ is called Hamming distance $\mathrm{d_H}(\mathbf{x}, \mathbf{y})$. The Hamming weight and the Hamming distance of a linear code C are coincided, and defined as the smallest Hamming weight of nonzero codewords of C:

$$\mathrm{d_H}(C) = \min\{\mathrm{wt_H}(\mathbf{x}) \mid \mathbf{x} \neq \mathbf{0},\ \mathbf{x} \in C\}.$$

[5, 6] proved that the Hamming distance of each λ-constacyclic code over

$\mathbb{F}_{p^m}$ is completely determined which is not depend on m. The Hamming distance of each λ-constacyclic code depends on the characteristic p of the finite field and the code length p^s.

Theorem 2.3. (cf. [5, 6]) *Let C be a λ-constacyclic code of length p^s over $\mathbb{F}_{p^m}$, then $C = \langle (x-\lambda_0)^i \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s}-\lambda \rangle}$, for $i \in \{0, 1, \ldots, p^s\}$, and its Hamming distance $\mathrm{d_H}(C)$ is determined by:*

$$\mathrm{d_H}(C) = \begin{cases} 1, \text{ if } i = 0 \\ (e+1)p^k, \\ \text{ if } p^s - p\cdot t + (e-1)t + 1 \leq i \leq p^s - p\cdot t + e\cdot t, \\ \text{ where } t = p^{s-k-1},\ 1 \leq e \leq p-1,\ 0 \leq k \leq s-1 \\ 0, \text{ if } i = p^s. \end{cases}$$

The ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} (u^2 = 0)$ can be expressed as $\mathcal{R} = \frac{\mathbb{F}_{p^m}[u]}{\langle u^2 \rangle} = \{a + ub \mid a, b \in \mathbb{F}_{p^m}\}$. It is easy to check that $\mathcal{R}$ is a chain ring with maximal ideal $u\mathbb{F}_{p^m}$. The ring $\mathcal{R}$ has precisely $p^m(p^m-1)$ units, which are of the forms $\alpha + u\beta$ and γ, where α, β, γ are nonzero elements of the field $\mathbb{F}_{p^m}$. In 2010, Dinh [6] gave the structure of all constacyclic codes of length p^s over $\mathcal{R}$ as follow.

Theorem 2.5. (cf. [6]) *Let λ be a unit of the ring $\mathcal{R}$, i.e., λ is of the form $\alpha + u\beta$ or γ, where α, β, γ are nonzero elements of the field $\mathbb{F}_{p^m}$.*

1) *If $\lambda = \alpha + u\beta$, then the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-(\alpha+u\beta) \rangle}$ is a chain ring with maximal ideal $\langle \alpha_0 x - 1 \rangle$, and $\langle (\alpha_0 x-1)^{p^s} \rangle = \langle u \rangle$. The $(\alpha+u\beta)$-constacyclic codes of length p^s over $\mathcal{R}$ are the ideals $\langle (\alpha_0 x-1)^i \rangle$, $0 \leq i \leq 2p^s$, of the chain ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-(\alpha+u\beta) \rangle}$.*

2) *If $\lambda = \gamma \in \mathbb{F}_{p^m}^*$, then the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma \rangle}$ is a local ring with the maximal ideal $\langle u, x-\gamma_0 \rangle$, but it is not a chain ring. The γ-constacyclic codes of length p^s over $\mathcal{R}$, i.e., ideals of the ring $\frac{\mathcal{R}[x]}{\langle x^{p^s}-\gamma \rangle}$, are*

   - *Type 1 (trivial ideals): $\langle 0 \rangle$, $\langle 1 \rangle$.*
   - *Type 2 (principal ideals with nonmonic polynomial generators): $\langle u(x-\gamma_0)^i \rangle$, where $0 \leq i \leq p^s - 1$,*
   - *Type 3 (principal ideals with monic polynomial generators): $\left\langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x) \right\rangle$, where $1 \leq i \leq p^s-1$, $0 \leq t < i$, and either $h(x)$ is 0 or $h(x)$ is a unit in $\mathbb{F}_{p^m}[x]$.*
   - *Type 4 (nonprincipal ideals): $\left\langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa \right\rangle$, with $h(x)$ as in Type 3, $\deg(h) \leq \kappa - t - 1$, and $\kappa < T$, where T is the smallest integer such that $u(x-\gamma_0)^T \in \left\langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x) \right\rangle$;*

i.e., such T can be determined as

$$T = \begin{cases} i, & \text{if } h(x) = 0 \\ \min\{i, p^s - i + t\}, & \text{if } h(x) \neq 0 \end{cases}.$$

3. Hamming distance

When the unit λ is of the form $\alpha + u\beta$, the Hamming distances of all $(\alpha + u\beta)$-constacyclic codes of length 5 over $\mathcal{R} = \mathbb{F}_{5^m} + u\mathbb{F}_{5^m}$ were provided in [6].

Theorem 3.1. (cf. [6]) *Let C be a $(\alpha+u\beta)$-constacyclic codes of length p^s over $\mathcal{R}$, then $C = \langle(\alpha_0 x - 1)^i\rangle \subseteq \mathcal{R}_{\alpha,\beta}$, for $i \in \{0, 1, \ldots, 2p^s\}$, and the Hamming distance $\mathrm{d_H}(C)$ is completely determined by*

$$\mathrm{d_H}(C) = \begin{cases} 1, \text{ if } \ 0 \leq i \leq p^s \\ (t+1)p^k, \\ \ \text{if } \ 2p^s - p \cdot r + (t-1)r + 1 \leq i \leq 2p^s - p \cdot r + t \cdot r \\ \ \text{where } r = p^{s-k-1}, \, 1 \leq t \leq p-1, \, 0 \leq k \leq s-1 \\ 0, \text{ if } \ i = 2p^s. \end{cases}$$

In this section, we compute the Hamming distances of the remaining λ-constacyclic codes, where $\lambda = \gamma \in \mathbb{F}_{5^m}^*$ as classified into 4 types in Theorem 2.5. Note that $\mathbb{F}_{5^m}$ is a subring of $\mathcal{R}$, for a code C over $\mathcal{R}$, we denote $\mathrm{d_H}(C_F)$ as the Hamming distance of $C\big|_{\mathbb{F}_{5^m}}$. For each codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ over $\mathcal{R}$, its polynomial representative $c(x)$ can be expressed as $c(x) = f(x) + ug(x)$, where $f(x), g(x) \in \mathbb{F}_{5^m}[x]$, with corresponding words $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$, $\mathbf{g} = (g_0, g_1, \ldots, g_{n-1})$ over $\mathbb{F}_{p^m}$. As $c_i = f_i + ug_i$, $c_i = 0$ if and only if $f_i = g_i = 0$, hence $\mathrm{wt_H}(c(x)) \geq \max\{\mathrm{wt_H}(f(x)), \mathrm{wt_H}(g(x))\}$.

Type 1 consists of the trivial ideals $\langle 0\rangle$, $\langle 1\rangle$, and obviously they have Hamming distances 0 and 1, respectively. For a code $C = \langle u(x-\gamma_0)^i\rangle$ of Type 2, $0 \leq i \leq p^s - 1$, the codewords of C are precisely the codewords of the γ-constacyclic codes $\langle(x-\gamma_0)^i\rangle$ in $\frac{\mathbb{F}_{5^m}[x]}{\langle x^5 - \gamma\rangle}$ multiplied by u. Therefore $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(x-\gamma_0)^i\rangle_F)$, which are given in Theorem 2.4.

Theorem 3.2. *Let $C = \langle u(x-\gamma_0)^i\rangle$, $0 \leq i \leq 4$, be a γ-constacyclic codes of length 5 over $\mathcal{R}$ of Type 2. Then $\mathrm{d_H}(C) = \mathrm{d_H}(\langle(x-\gamma_0)^i\rangle_F)$, and is determined by*

$$\mathrm{d_H}(C) = \begin{cases} 1, \ \textit{if} \ \ i = 0 \\ (e+1)p^k, \\ \ \textit{if } 5 - 5 \cdot r + (t-1)r + 1 \leq i \leq 5 - 5 \cdot r + e \cdot r \\ \textit{where } r = 5^{-k}, \, 1 \leq e \leq 4, \ \ \textit{and} \ \ 0 \leq k \end{cases}$$

Theorem 3.3. *Let C be a γ-constacyclic codes of length 5 over $\mathcal{R}$ of Type 3, i.e., $C = \langle (x-\gamma_0)^i + u(x-\lambda_0)^t h(x)\rangle$, where $1 \le i \le 4$, $0 \le t < i$, and either $h(x)$ is 0 or $h(x)$ is a unit in $\mathbb{F}_{5^m}[x]$. Then $\mathrm{d_H}(C) = \mathrm{d_H}(\langle (x-\gamma_0)^i\rangle_F)$, and is determined by*

$$\mathrm{d_H}(C) = (e+1)5^k,$$

where $5 - 5^{1-k} + (t-1)5^{1-k-1} + 1 \le i \le 5 - 5^{1-k} + e5^{-k}$, $1 \le e \le 4$, and $0 \le k$.

Proof. First of all, since $u(x-\gamma_0)^i = u[(x-\gamma_0)^i + u(x-\lambda_0)^t h(x)] \in C$, it follows that

$$\mathrm{d_H}(C) \le \mathrm{d_H}(\langle u(x-\gamma_0)^i\rangle) = \mathrm{d_H}(\langle (x-\gamma_0)^i\rangle_F).$$

Now, consider an arbitrary polynomial $c(x) \in C$. That means there exist $f_0(x), f_u(x) \in \mathbb{F}_{5^m}[x]$ such that

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x-\gamma_0)^i + u(x-\lambda_0)^t h(x)] \\ &= f_0(x)(x-\gamma_0)^i + u[f_0(x)(x-\lambda_0)^t h(x) + f_u(x)(x-\gamma_0)^i]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{wt_H}(c(x)) &\ge \max\left\{\mathrm{wt_H}(f_0(x)(x-\gamma_0)^i), \mathrm{wt_H}(r(x))\right\} \\ &\ge \max\left\{\mathrm{wt_H}(f_0(x)(x-\gamma_0)^i), \mathrm{wt_H}(f_u(x)(x-\gamma_0)^i)\right\} \\ &\ge \mathrm{d_H}(\langle (x-\gamma_0)^i\rangle_F), \end{aligned}$$

where $r(x) = f_0(x)(x-\lambda_0)^t h(x) + f_u(x)(x-\gamma_0)^i$. Hence, $\mathrm{d_H}(\langle (x-\gamma_0)^i\rangle_F) \le \mathrm{d_H}(C)$, forcing $\mathrm{d_H}(\langle (x-\gamma_0)^i\rangle_F) = \mathrm{d_H}(C)$. $\square$

Theorem 3.4. *Let C be a γ-constacyclic codes of length 5 over $\mathcal{R}$ of Type 4, i.e., $C = \langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa\rangle$, with $h(x)$ as in Type 3, $\deg(h) \le \kappa - t - 1$, and $\kappa < T$, where T is the smallest integer such that $u(x-\gamma_0)^T \in \left\langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x)\right\rangle$; i.e., such T can be determined as*

$$T = \begin{cases} i, & \text{if } h(x) = 0 \\ \min\{i, 5-i+t\}, & \text{if } h(x) \ne 0\ . \end{cases}$$

Then $\mathrm{d_H}(C) = \mathrm{d_H}(\langle (x-\gamma_0)^\kappa\rangle_F)$, and is determined by

$$\mathrm{d_H}(C) = (e+1)5^k,$$

where $5 - p^{1-k} + (t-1)5^{-k} + 1 \le \kappa \le 5^s - 5^{s-k} + e5^{-k}$, $1 \le e \le 4$, and $0 \le k$.

Proof. Clearly, $C = \langle (x-\gamma_0)^i + u(x-\gamma_0)^t h(x), u(x-\gamma_0)^\kappa\rangle \supseteq \langle u(x-\gamma_0)^\kappa\rangle \supseteq \langle u(x-\gamma_0)^i\rangle$, since $\kappa < T \le i$. Thus, $\mathrm{d_H}(C) \le \mathrm{d_H}(\langle u(x-\gamma_0)^i\rangle) = \mathrm{d_H}(\langle (x -$

$\gamma_0)^i\rangle_F)$. To prove that $\mathrm{d_H}(\langle(x-\gamma_0)^i\rangle_F) \leq \mathrm{d_H}(C)$, we consider an arbitrary polynomial $c(x) \in C$ and proceed to show that $\mathrm{wt_H}(c(x)) \geq \mathrm{d_H}(\langle(x-\gamma_0)^i\rangle_F)$. Now there are $f_0(x), f_u(x), g_0(x), g_u(x) \in \mathbb{F}_{p^m}[x]$ such that

$$\begin{aligned}
c(x) &= [f_0(x)+uf_u(x)][(x-\gamma_0)^i + u(x-\lambda_0)^t h(x)] \\
&\quad + u(x-\gamma_0)^\kappa[g_0(x)+ug_u(x)] \\
&= f_0(x)(x-\gamma_0)^i + u[f_0(x)(x-\lambda_0)^t h(x) \\
&\quad + f_u(x)(x-\gamma_0)^i + g_0(x)(x-\gamma_0)^\kappa] \\
&= f_0'(x)(x-\gamma_0)^\kappa + u[f_0(x)(x-\lambda_0)^t h(x) + g_0'(x)(x-\gamma_0)^\kappa],
\end{aligned}$$

where $f_0'(x) = f_0(x)(x-\gamma_0)^{i-\kappa} \in \mathbb{F}_{p^m}[x], g_0'(x) = f_u(x)(x-\gamma_0)^{i-\kappa} + g_0(x) \in \mathbb{F}_{p^m}[x]$. Hence,

$$\begin{aligned}
\mathrm{wt_H}(c(x)) &\geq \max\{\mathrm{wt_H}(f_0'(x)(x-\gamma_0)^\kappa), \mathrm{wt_H}(r'(x))\} \\
&\geq \max\{\mathrm{wt_H}(f_0'(x)(x-\gamma_0)^\kappa), \mathrm{wt_H}(g_0'(x)(x-\gamma_0)^\kappa)\} \\
&\geq \mathrm{d_H}(\langle(x-\gamma_0)^i\rangle_F),
\end{aligned}$$

where $r'(x) = f_0(x)(x-\lambda_0)^t h(x) + g_0'(x)(x-\gamma_0)^\kappa$. $\square$

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