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THE APPROXIMATE SOLUTION OF STOCHASTIC VAN DER POL - DUFFING SYSTEM WITH TIME DELAY BY SECOND ORDER STOCHASTIC AVERAGING METHOD

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Abstract

The paper shows that the approximate solution of the Van der Pol-Duffing system with time delay subjected to the white noise can be found by the second order stochastic averaging method. The stochastic system with time delay is transformed into the stochastic non-delay equation in Ito sense in accordance with the hypothesis that there are some slowly varying processes. Then the higher order stochastic averaging method is artfully applied to find the stationary probability density function for the system. The analytical results are verified by numerical simulation results.

Key words: Stochastic system, stochastic averaging method, time delay, slowly varying processes.

1 Introduction

Throughout the century, there has been a growing interest in the effects of noise on dynamical systems with delays which are considered as stochastic delay differential equations. Especially, in science and engineering we often experience non-linear systems subjected to random excitations. In the case of no-time delay, the stochastic averaging method, a versatile and powerful approximate approach to random vibration problems of non-linear systems, is strongly introduced. Originally initiated by Krylov and Bogoliubov and then developed by Mitropolskii and extended as the higher order averaging method by Anh [1,2,3,4], the stochastic averaging method was first extended by Stratonovich in the field of random vibrations and mathematically tested by Khasminskii [5,6]. By applying the stochastic averaging method, random systems can be described in averaged Ito stochastic differential equations, whose solutions are Markov processes. The stochastic averaging method is also applied in Cartesian coordinates by Anh et al [7] and Hao et al [8].

In the field of nonlinear oscillators, the Van der Pol-Duffing system is one of the classes of systems that has caught researchers attentions. The basic method to study this kind of system is the averaging method. Mitropolskii et al [9,10] found the stationary density of the solution by the stochastic averaging method. Anh [4] and Tinh [11] further worked out the stationary density of the solution in the higher order approximation. Zhu et al [12] and Kumar et al [13], in practice, also applied a new stochastic averaging method to predict the response of a Van der Pol- Duffing oscillator under both external and parametric excitation of wide-band stationary random processes. Xu et al [14] investigated the stability and control of two-dof coupled Duffing-Van der Pol systems under stochastic Gaussian excitations.

In recent years, the Van der Pol oscillator with delayed feedback control has attracted many researches. Mittropolskii et al [9] proposed approximate stochastic systems for stochastic systems with delayed control feedback and proved the theorems for this approximate method. Maccari [15] demonstrated that if the vibration control terms are added, stable periodic solution with arbitrarily chosen amplitude can be accomplished. Atay [16] investigated the effect of delayed feedback on the classical van der Pol oscillator. Liu et al [17] applied the stochastic averaging method for quasi-integrable Hamiltonian systems for a Duffing Van der Pol oscillator with delayed linear feedback control subject to additive Gaussian white noise excitation. Jin et al [18] proposed the stochastic averaging method to investigate the response and stability for the dynamic system with delayed feedback control and additive or multiplicative Gaussian white noise. Hao and Anh [19] investigated the stationary probability density functions of the Duffing oscillator with time delay subjected to combined harmonic and random excitation by the method of stochastic averaging and equivalent linearization.

In this paper, we consider the single degree of freedom (SDOF) Van der Pol-Duffing system with time delay under additive white noise excitation

$$\ddot{x} + \omega^2 x - \varepsilon\left[\left(\alpha - \beta x^2\right)\dot{x} + \gamma x^3 + \varsigma x\left(t - \Delta\right) + \zeta \dot{x}\left(t - \Delta\right)\right] = \sqrt{\varepsilon}\sigma\xi_t \qquad (1)$$

where $\omega > 0, \gamma > 0, \alpha > 0,\ \beta > 0,\ \sigma > 0, \varsigma,\ \zeta$ are constants, ε and Δ are small positive parameters, ξ_t is a Gaussian white noise process of unit intensity,

$$E\left[\xi_t\right] = 0,\ \ E\left[\xi_t \xi_{t+\tau}\right] = \delta\left(\tau\right) \qquad (2)$$

where $E\left(\cdot\right)$ denotes the mathematic expectation operator.
We study the response of the system (1) by the stochastic averaging method and figure out the analytical expression of stationary probability density function in higher approximation. Hence, the delay terms can be expressed in terms of the system states without time delay and harmonic functions of delay first, and then the system is transformed into Ito stochastic differential equations without time delay, from which the averaged Ito equations are derived. In Section 2, the stochastic averaging method is used to determine the averaged Ito stochastic differential equations of the system for the general case; In section 3, we show the response of the Van der Pol- Duffing system by applying the higher order stochastic averaging method. Section 4 show numerical results. The paper closes with some conclusions in section 5.

2 The stochastic averaging method

Let us study the SDOF system with time delay described by the autonomous equation

$$\frac{d^2x\left(t\right)}{dt^2} + \omega^2 x = \varepsilon f\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right) + \sqrt{\varepsilon} g\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right)\xi_t \qquad (3)$$

where ξ_t is a white noise of unit intensity, ε and Δ are small positive parameter, $\omega > 0$ is a constant, $x_\Delta = x\left(t - \Delta\right)$, $\dot{x}_\Delta = \dot{x}\left(t - \Delta\right)$. Assuming that equation (3) has a stationary response. Equation (3) may be considered as the following system of stochastic differential equations in the Ito sense

$$\begin{aligned} dx &= \dot{x}dt \\ d\dot{x} &= \left[-\omega^2 x + \varepsilon f\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right)\right]dt + \sqrt{\varepsilon} g\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right)dB_t \end{aligned} \qquad (4)$$

where B_t is the standard Brownian motion. When $\varepsilon = 0$, the equation (3) has the periodic solution

$$x = a\left(t\right)\cos\varphi, \quad \dot{x} = -a\left(t\right)\omega\sin\varphi, \quad \varphi = \omega t + \theta\left(t\right) \qquad (5)$$

In order to apply the stochastic averaging method, we transform $(x, \dot{x})$ to (a, θ) by the change (5). Applying Ito's rule, we can rewrite the system (4) as follows (see [2])

$$
\begin{aligned}
da &= \varepsilon\left[\frac{1}{2a\omega^2}g^2\cos^2\varphi - \frac{1}{\omega}f\sin\varphi\right]dt - \frac{\sqrt{\varepsilon}}{\omega}g\sin\varphi dB_t,\\
d\theta &= \varepsilon\left[-\frac{1}{2a\omega^2}g^2\sin 2\varphi - \frac{1}{a\omega}f\cos\varphi\right]dt - \frac{\sqrt{\varepsilon}}{a\omega}g\cos\varphi dB_t,
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
& f = f\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right),\\
& g = g\left(x, \dot{x}, x_\Delta, \dot{x}_\Delta\right),\\
& x = a\cos\varphi,\ \dot{x} = -a\omega\sin\varphi,\\
& x_\Delta = a\left(t-\Delta\right)\cos\left[\omega\left(t-\Delta\right)+\theta\left(t-\Delta\right)\right],\\
& \dot{x}_\Delta = -a\left(t-\Delta\right)\omega\sin\left[\omega\left(t-\Delta\right)+\theta\left(t-\Delta\right)\right],
\end{aligned} \tag{7}
$$

and B_t is a standard Brownian motion.
From (6), we can see that $\dot{a}$ and $\dot{\theta}$ depend on the small parameter ε so we can assume that $a\left(t\right)$ and $\theta\left(t\right)$ are slowly varying processes while the average value of the instantaneous phase is a fast varying process. Thus, over one period $T = \frac{2\pi}{\omega}$ we can replace $a\left(t-\Delta\right)$ and $\theta\left(t-\Delta\right)$ by $a\left(t\right)$ and $\theta\left(t\right)$, respectively. The pair (a, θ) is an approximate two-dimensional diffusion process.
The Fokker-Planck (FP) equation for the density function of probability $W\left(a, \theta, t\right)$ of system (6) is:

$$
\begin{aligned}
\frac{\partial W}{\partial t} =& \varepsilon\left\{\frac{\partial}{\partial a}\left(K_1W\right) + \frac{\partial}{\partial\theta}\left(K_2W\right)\right\}\\
& \varepsilon\left\{-\frac{1}{2}\left[\frac{\partial^2}{\partial a^2}\left(K_{11}W\right) + 2\frac{\partial^2}{\partial a\partial\theta}\left(K_{12}W\right) + \frac{\partial^2}{\partial\theta^2}\left(K_{22}W\right)\right]\right\}
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
K_1\left(a, \varphi\right) &= \frac{1}{2a\omega^2}g^2\cos^2\varphi - \frac{1}{\omega}f\sin\varphi,\\
K_2\left(a, \varphi\right) &= -\frac{1}{2a\omega^2}g^2\sin 2\varphi - \frac{1}{a\omega}f\cos\varphi,\\
K_{11}\left(a, \varphi\right) &= \frac{1}{\omega^2}\sin^2\varphi g^2\left(a, \varphi\right),\\
K_{12}\left(a, \varphi\right) &= \frac{\cos\varphi\sin\varphi}{a\omega^2}g^2\left(a, \varphi\right),\\
K_{22}\left(a, \varphi\right) &= \frac{\cos^2\varphi}{a^2\omega^2}g^2\left(a, \varphi\right).
\end{aligned} \tag{9}
$$

As known, solving the FP equation is a difficult problem. However, this problem is essentially simplified by using the averaging method. Averaging the FP equation (8) we obtain the averaged equation FP for the probability density function:

$$\begin{aligned}\frac{\partial W}{\partial t} &= \varepsilon\left\{\frac{\partial}{\partial a}\left(\bar{K}_1 W\right)+\frac{\partial}{\partial \theta}\left(\bar{K}_2 W\right)\right\} \\ &-\varepsilon\left\{\frac{1}{2}\left[\frac{\partial^2}{\partial a^2}\left(\bar{K}_{11} W\right)+2\frac{\partial^2}{\partial a \partial \theta}\left(\bar{K}_{12} W\right)+\frac{\partial^2}{\partial \theta^2}\left(\bar{K}_{22} W\right)\right]\right\}\end{aligned} \tag{10}$$

where the top line denotes the averaging in t.

$$\bar{f}=\frac{1}{2\pi}\int_0^{2\pi/\omega} f dt=\frac{1}{2\pi}\int_0^{2\pi} f d\varphi, \tag{11}$$

If the stationary density $W(a,\theta)$ exists then it satisfies the equation

$$\frac{\partial}{\partial a}\left(\bar{K}_1 W\right)+\frac{\partial}{\partial \theta}\left(\bar{K}_2 W\right)=\frac{1}{2}\left[\frac{\partial^2}{\partial a^2}\left(\bar{K}_{11} W\right)+2\frac{\partial^2}{\partial a \partial \theta}\left(\bar{K}_{12} W\right)+\frac{\partial^2}{\partial \theta^2}\left(\bar{K}_{22} W\right)\right] \tag{12}$$

The solution of this equation should be non-negative and normalized. Since the considered equation (3) is autonomous, the averaged FP equation (10), written for the probability density function $W(a,\theta)$, if exists, corresponding to the system (6), has the form

$$\frac{\partial W}{\partial t}=\varepsilon\left[\frac{\partial}{\partial a}\left(\bar{K}_1(a) W\right)-\frac{1}{2}\frac{\partial^2}{\partial a^2}\left(\bar{K}_{11}(a) W\right)\right] \tag{13}$$

Therefore equation (12) becomes

$$\frac{\partial}{\partial a}\left(\bar{K}_1(a) W\right)=\frac{1}{2}\frac{\partial^2}{\partial a^2}\left(\bar{K}_{11}(a) W\right) \tag{14}$$

with the initial condition is $W\left(a,t\,|\,a_0,t_0\right)$. Solving (14) one gives the solution

$$W(a)=\frac{C}{\bar{K}_{11}(a)}\exp\left\{\int\frac{2\bar{K}_1(a)}{\bar{K}_{11}(a)}da\right\} \tag{15}$$

where C is a normalization constant determined from the condition

$$\int_0^{2\pi}\int_0^{\infty} W(a,\varphi)\, da d\varphi=1 \tag{16}$$

It is know that in many cases of interest the averaged FP equation (13) is not sufficient for analysis of nonlinear terms in the original equation (3). Anh [4] extended the classical averaging method to get the effect of nonlinear terms on the stationary solution of the considered systems. Tinh [11] (pp. 59-60) showed that higher order approximate solution is more accurate than first-order one. For the given functions $K_j(a,\varphi), K_{ij}(a,\varphi), i,j=1,2$, we define two operators as follows

$$\begin{aligned}
[K_i, K_{ij}]\, L(W) = {} & \frac{\partial}{\partial a}(K_1 W) + \frac{\partial}{\partial \varphi}(K_2 W) \\
& - \frac{1}{2}\left[\frac{\partial^2}{\partial a^2}(K_{11}W) + \frac{2\partial^2}{\partial a \partial \varphi}(K_{12}W) + \frac{\partial^2}{\partial \varphi^2}(K_{22}W)\right] \\
[K_i, K_{ij}]\, \ell(W) = {} & \frac{\partial K_1}{\partial a} + \frac{\partial K_2}{\partial \varphi} - \frac{1}{2}\frac{\partial^2 K_{11}}{\partial a^2} - \frac{\partial K_{12}}{\partial a \partial \varphi} - \frac{1}{2}\frac{\partial^2 K_{22}}{\partial \varphi^2} \\
& + \left(K_1 - \frac{\partial K_{11}}{\partial a} - \frac{\partial K_{12}}{\partial \varphi}\right)\frac{\partial W}{\partial a} + \left(K_2 - \frac{\partial K_{22}}{\partial \varphi} - \frac{\partial K_{12}}{\partial a}\right)\frac{\partial W}{\partial \varphi} \\
& - \frac{1}{2}\left\{K_{11}\left(\frac{\partial^2 W}{\partial a^2} + \left(\frac{\partial W}{\partial a}\right)^2\right) + 2K_{12}\left(\frac{\partial^2 W}{\partial a \partial \varphi} + \frac{\partial W}{\partial a}\frac{\partial W}{\partial \varphi}\right)\right. \\
& \left. + K_{22}\left(\frac{\partial^2 W}{\partial \varphi^2} + \left(\frac{\partial W}{\partial \varphi}\right)^2\right)\right\}.
\end{aligned}$$

Hence, the equation (8) can be written in the form

$$\omega \frac{\partial W}{\partial \varphi} = -\varepsilon\, [K_i, K_{ij}]\, L(W) \tag{17}$$

Given (3) is autonomous. Suppose that

$$K_i(a,\varphi,\varepsilon) = K_{io}(a,\varphi) + R_{i1}(a,\varphi)\varepsilon + R_{i2}(a,\varphi)\varepsilon^2 + \cdots,\ i = 1,2 \tag{18}$$

Then the approximate solution of (8) can be found in the form of

$$W \approx W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + ... \tag{19}$$

where

$$W_0(a) = \frac{C}{\bar{K}_{11}} \exp\left(\int 2\frac{\bar{K}_{10}}{\bar{K}_{11}} da\right) \tag{20}$$

$$W_1(a,\varphi) = -\frac{1}{\omega}\int [K_{i0}, K_{ij}]\, L(W_0)\, d\varphi = W_0(a)\,(W_{10}(a) + W_{11}(a,\varphi)) \tag{21}$$

$$\begin{aligned} W_2(a,\varphi) &= -\frac{1}{\omega}\int \left\{[K_{i0}, K_{ij}]\, L(W_1) - [R_{i1}, 0]\, L(W_0)\right\} d\varphi \\ &= W_0(a)\,(W_{20}(a) + W_{22}(a,\varphi)) \end{aligned} \tag{22}$$

$$W_n(a,\varphi) = -\frac{1}{\omega}\int \left\{[K_{i0}, K_{ij}]\, L(W_{n-1}) - \sum_{\ell=1}^{n-1} [R_{i\ell}, 0]\, L(W_{n-1-\ell})\right\} d\varphi,\ n \geq 3 \tag{23}$$

where

$$\begin{aligned} W_{11}(a,\varphi) = -\frac{2}{\omega}\sum_{n=1}^{\infty}\frac{1}{n}\Big\{&\overline{[K_{i0}, K_{ij}]\,\ell(\ln W_0(a))\cos n\varphi}\sin n\varphi - \\ &\overline{[K_{i0}, K_{ij}]\,\ell(\ln W_0(a))\sin n\varphi}\cos n\varphi\Big\}, \end{aligned} \tag{24}$$

$$\begin{aligned} W_{10}(a) = \int \Bigg[&\frac{2\overline{K_{10}W_{11}}}{\bar{K}_{11}} - \frac{\partial}{\partial a}\overline{K_{11}W_{11}} \\ &- \overline{K_{11}W_{11}}\frac{1}{W_0(a)}\frac{\partial W_0(a)}{\partial a} - 2\frac{\partial \bar{R}_{11}}{\partial a} - 2\bar{R}_{11}\frac{1}{W_0(a)}\frac{\partial W_0(a)}{\partial a}\Bigg] da, \end{aligned} \tag{25}$$

From (19) to (25), we have the second order approximate stationary solution of (8) in the form of

$$W = W_0(a)\left\{1 + \varepsilon\,(W_{10}(a) + W_{11}(a,\varphi))\right\}. \tag{26}$$

3 The approximate solution of the stochastic Van der Pol-Duffing system with time delay

Here we apply the stochastic averaging method in the preceding section on studying the response of Van der Pol- Duffing system

$$\ddot{x} + \omega^2 x - \varepsilon\left[\left(\alpha - \beta x^2\right)\dot{x} - \gamma x^3 + \varsigma x_\Delta + \zeta \dot{x}_\Delta\right] = \sqrt{\varepsilon}\sigma\xi_t \tag{27}$$

where $\alpha > 0$, $\beta > 0$, $\omega \neq 0$, γ, ς, ζ, Δ are constants, ε is a small positive parameter, ξ_t is white noise, $x_\Delta = x(t-\Delta)$, $\dot{x}_\Delta = \dot{x}(t-\Delta)$. From (7) and

(9) we have

$$\begin{aligned}
\bar{K}_1 &= -\frac{1}{8}a^3\beta + \frac{\kappa a}{2} + \frac{\sigma^2}{4a\omega^2}, \\
\bar{K}_2 &= \frac{3a^2\gamma}{8\omega} - \frac{\eta}{2\omega}, \\
\bar{K}_{11} &= \frac{\sigma^2}{2\omega^2}, \ \bar{K}_{12} = 0, \ \ \bar{K}_{22} = \frac{\sigma^2}{2a^2\omega^2},
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\eta &= \varsigma \cos\omega\Delta + \omega\zeta \sin\omega\Delta \\
\kappa &= \alpha + \zeta \cos\omega\Delta - \varsigma\omega^{-1}\sin\omega\Delta.
\end{aligned} \tag{29}$$

It is clear that the averaged FP equation (13) for the stationary density, if exists, is a simple form

$$\frac{\partial}{\partial a}\left(\bar{K}_1 W\right) = \frac{1}{2}\frac{\partial^2}{\partial a^2}\left(\bar{K}_{11} W\right), \tag{30}$$

Since $\bar{K}_{11}$ does not depend on a so, from (20), we can seek the first order approximate solution for probability density for (27) as follows

$$W_0\left(a\right) = C_0 \exp\int \frac{2\bar{K}_1}{\bar{K}_{11}}da = C_0 a \exp\left\{\frac{\kappa\omega^2 a^2}{\sigma^2} - \frac{\omega^2 a^4\beta}{8\sigma^2}\right\} \tag{31}$$

where C_0 determined from (16) is

$$C_0 = \frac{\omega\sqrt{2\beta}}{2\pi\sqrt{\pi}\sigma}\exp\left(-\frac{2\omega^2\kappa^2}{\sigma^2\beta}\right)\left[1 + \operatorname{erf}\left(\frac{\omega\kappa\sqrt{2}}{\sigma\sqrt{\beta}}\right)\right]^{-1} \tag{32}$$

in which, erf (x) is the error function defined by

$$\operatorname{erf}\left(x\right) = \frac{2}{\sqrt{\pi}}\int\limits_0^x \exp\left(-t^2\right)dt \tag{33}$$

It is seen that the solution (31) corresponds to the case $\gamma = 0$ in (27) and does not include the effect of the nonlinear term γx^3. In this approximation, solution (31) can be seen as a special case in [18] (p. 337). Now we seek the solution of (17) in the form of (19). In order to find the second approximate solution of (17), we apply the higher order stochastic averaging procedure. Calculating from (24) and (25) gives

$$W_{11}(a,\varphi)=\frac{\omega\beta a^4\sin 4\varphi}{64\sigma^2}\left(\beta a^2-4\kappa\right)-\frac{\beta a^2\sin 2\varphi}{32\omega\sigma^2}\left(\beta\omega^2a^4-12\sigma^2-4\omega^2\kappa a^2\right)$$
$$-\frac{\gamma a^4\cos 4\varphi}{64\sigma^2}\left(a^2\beta-4\kappa\right)-\frac{a^2\cos 2\varphi}{16\sigma^2}\left(\beta a^2-4\kappa\right)\left(\gamma a^2-2\eta\right), \quad (34)$$

$$W_{10}(a)=-\frac{a^2}{96\sigma^2}\left(a^4\gamma\beta-18a^2\gamma\kappa+48\eta\kappa\right) \quad (35)$$

Thus, the second order approximate solution of (27) is

$$W_1(a,\varphi)=C_1a\exp\left(\frac{\kappa\omega^2a^2}{\sigma^2}-\frac{\omega^2a^4\beta}{8\sigma^2}\right)\{1+\varepsilon\left[W_{10}(a)+W_{11}(a,\varphi)\right]\} \quad (36)$$

where C_1 is a normalization constant.
We can see that if $\beta=\varsigma=\zeta=0$ then (36) becomes the solution of the equation in [2] (pp 384-385). In this situation, the nonlinear γx^3 reduces the mean square of the amplitude [4]. In general, the probability density function at the second order approximation for the amplitude is

$$W(a)=\int_0^{2\pi}W_1(a,\varphi)\,d\varphi$$
$$=\frac{C_1\pi a}{32\sigma^2}\exp\left(-\frac{\omega^2\beta}{8\sigma^2}a^4+\frac{\kappa\omega^2}{\sigma^2}a^2\right).$$
$$\left\{64\sigma^2-\varepsilon\left(8\eta\beta a^4+12\gamma\kappa a^4-3\gamma\beta a^6+32\eta\kappa a^2\right)\right\} \quad (37)$$

s Hence, using the second approximate solution, the effect of the nonlinear term γx^3 and delay parameter is obtained in the formulae from (36) to (37).

4 Numerical results

The mean-square response of equation (27) is obtained by

$$E_i\left[x^2\right]=\int_0^{2\pi}\int_0^{\infty}a^2\cos^2\varphi\;W_i(a,\varphi)\,da\,d\varphi, i=0,1 \quad (38)$$

where $W_i(a,\varphi)$ is the stationary density function of (27). In the table below, we consider the case where $\varepsilon=0.01$, $\alpha=1$, $\beta=5$, $\omega=1$, $\gamma=2$, $\varsigma=1$, $\zeta=1$, $\sigma=1$.

		1st approximation		2nd approximation	
Δ	Monte-Carlo simulation	$E_0\left(x^2\right)$	Error	$E_1\left(x^2\right)$	Error
0	0.8226	0.8374	1.798	0.8149	0.9370
0.1	0.79008	0.8025	1.5720	0.7815	1.0860
0.2	0.7452	0.7656	2.7350	0.7461	0.1180

Table: The effect of delay parameter Δ on the mean-square response of the Van der pol- Duffing oscillator obtained by Monte-Carlo simulation and by higher-order stochastic averaging method.

The table shows that the relative error of the second order stochastic averaging approximation is smaller than the error of the first order stochastic averaging method when the delay parameter is small.

5 Conclusion

In the paper, the higher order stochastic averaging method is proposed to investigate the response for the stochastic Van der Pol- Duffing system with time delay under additive Gaussian white noise. The analytical solution in second order approximation of the stochastic Van der Pol- Duffing system is found for the first time. This solution shows the effect of the nonlinear term γx^3 which cannot be found by the original stochastic averaging method. Comparison results in the table above show that the higher order stochastic averaging method is useful in investigating nonlinear systems with time delay under a random excitation.

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