Southeast-Asian J. of Sciences Vol. 3, No. 1 (2014) pp. 24-32

OPTIMALITY CONDITIONS IN QUASI-CONVEX VECTOR OPTIMIZATION

Ho T. Quyen

Department of Mathematics
HCMC University of Architecture
Hochiminh City, Vietnam
e-mail: thucquyen911@gmail.com

Abstract

An optimization problem is quasiconvex if the objective is quasiconvex and the constraint set is convex. In this paper, we generalize some notion of quasiconvex scalar optimization to quasiconvex vector optimization and study optimality conditions. Our results contain and improve some recent ones in the literature.

1 Introduction

In vector optimization one investigates optimal elements such as minimal, strongly minimal, properly minimal or weakly minimal elements of a nonempty subset of a partially ordered linear space. The problem of determining at least one of these optimal elements, if they exist at all, is also called a vector optimization problem. Problems of this type can be found not only in mathematics but also in engineering and economics.

It is the purpose of this paper to present some optimality conditions for constrained optimization problems with quasiconvex functions, i.e., functions whose sublevel sets are convex. Such functions form the main class of generalized convex functions and are widely used in mathematical economics. There are a lot of papers dealing with optimality conditions for constrained problems under generalized convexity conditions (see [5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 18, 19, 20, 34] for example). In particular, for quasiconvex problem, we refer the reader to [1, 13, 14, 16, 21, 24, 26, 27, 28, 29, 30, 31].

Key words: Optimality conditions; Convex sublevel sets; Normal cone; Star subdifferentials.

However, we can observe that optimality conditions do not need sublevel sets being convex for all points in feasible sets. Employ this fact, in a recent paper [22], the authors study optimality conditions for minimization problems involving Plastria (or Gutiérrez) functions at a point x, i.e. its sublevel sets are convex and the normal cone at x to the sublevel set is generated by the Plastria (Gutiérrez) subdifferential.

But there are differentiable quasiconvex functions with the Plastria (Gutiérrez) subdifferentials which are empty at each point. So, it may be quite strict to be a Plastria (Gutiérrez) function at a point. Motivated by this fact, in [36] we study a more general class of functions with sublevel sets being convex at some points in feasible sets. We do not assume that data of problems are smooth. Thus, we use normal cones to the corresponding sublevel sets, which are Penot's variants of Greenberg-Pierskalla's subdifferential introduced in [23, 33], to present our results. We compare our results with some recent ones using adapted subdifferentials of quasiconvex analysis like the Plastria subdifferential [35], the infradifferential Gutiérrez subdifferential [10], and the Greenberg-Pierskalla's subdifferential [9].

We do not make a comparison with results using the all-purpose subdifferentials of nonsmooth analysis (see [3, 23]). The reason is that these subdifferentials are local, whereas the ones we use are of global characters; intermediate notions are presented in [4, 21], and [25]. Our optimality conditions generalize conditions in [9, 22, 26] using normal cones or subdifferentials related to normal cones to sublevel sets.

In this paper, we generalize our result in [36] for vector optimization context. The organization of this paper is as follows. Section 2 contains definitions and preliminaries needed in the sequel. Section 3 is devoted to optimality conditions for a minimization problem with a convex constraint set. In section 4 optimality conditions are established for the mathematical programming problem.

2 Preliminaries

Throughout the paper, let X be a normed space. For $A \subseteq X$ $\mathrm{int}A$, $\mathrm{cl}A$, $\mathrm{co}A$ and $\mathrm{cone}A$ denote the interior, closure, convex hull of A and the conical hull (called also the cone generated by A), i.e. $\mathrm{cone}A := \{\lambda x : x \in A, \lambda \in \mathbb{R}_+\}$, respectively. The distance from $x \in X$ to A is $\mathrm{dist}(x, A) = \inf\{\|x-y\| : y \in A\}$. X^* is the topological dual of X and $\langle ., . \rangle$ is the duality pairing. The normal cone at x to A, denoted by $N(A, x)$, is defined by

$$N(A, x) := \{x^* \in X^* : \forall u \in A, \langle x^*, u - x\rangle \leq 0\}.$$

If $x \notin A$ we adopt that $N(A, x) = \emptyset$. The contingent cone of A at $x \in X$, denoted by $T(A, x)$, is the following cone

$$T(A, x) := \{v \in X : \exists (r_n) \to 0_+, \exists (v_n) \to v, \forall n, x + r_n v_n \in A\}.$$

To see relationships between $N(A,x)$ and $T(A,x)$, recall that the polar cones of cones $B \subseteq X$ and $D \subseteq X^*$ are

$$B^- := \{x^* \in X^* : \forall x \in B, \langle x^*, x\rangle \leq 0\},$$

$$D^- := \{x \in X : \forall x^* \in D, \langle x^*, x\rangle \leq 0\}.$$

Clearly $N(A,x) = [\text{clcone}(A-x)]^-$. Setting, in the definition of $T(A,x)$, $x_n = x + r_n v_n$, we see that

$$T(A,x) = \{v : \exists (r_n) \to 0, \exists (x_n) \subseteq A \to x,\ \lim \frac{x_n - x}{r_n} = v\} \subseteq \text{clcone}(A-x).$$

Hence, $T(A,x)^- \supseteq N(A,x)$. Furthermore, if $v \in T(A,x)$, i.e. v is of the form $\lim \frac{x_n - x}{r_n}$, and $x^* \in N(A,x)$, then $\langle x^*, v\rangle \leq 0$. Therefore, $T(A,x) \subseteq N(A,x)^-$. Moreover, if A is convex then the above containments become equalities. $A \subseteq X$ is called strictly convex at $\bar{x}$ if $\langle x^*, x - \bar{x}\rangle < 0$ for every $x \in A \setminus \{\bar{x}\}$ and $x^* \in N(A,\bar{x}) \setminus \{0\}$. If $N(A,\bar{x}) \setminus \{0\} \neq \emptyset$, this strict convexity implies that $\bar{x}$ is an extreme point of A. The converse is not true.

Let K be a convex, pointed cone with apex at 0 and nonempty interior in a norm space Y . The order generated by K is given by

$$x \geq_K y \text{ if and only if } x - y \in K,$$

for $x, y \in Y$. A function f from a normed space X to Y is called *K-quasiconvex* if its sublevel set $L_f(x) := \{u \in X : f(u) \leq_K f(x)\}$ at x is convex for all $x \in X$. Another equivalent statement, which is often met in the literature, is that f is K-quasiconvex if for all $x, y \in \text{dom} f := \{x \in X : f(x) < +\infty\}$, all $t \in [0,1]$, $f((1-t)x + ty) \leq_K \max\{f(x), f(y)\}$ in the case K is a subset of R^m and generated by m linear independent vectors.

Consider the following set-constrained multiobjective problem:

$$\min\nolimits_K f(x), \text{ s.t. } x \in C, \tag{P}$$

where $f : X \longmapsto Y$ and C is a subset of X.

Definition 1. *The feasible point $\bar{x} \in S$ is called* efficient *or* Pareto optimal*, if there is no other $x \in C$ such that $f(x) \leq_K f(\bar{x})$.*

We consider the following normal-cone subdifferentials

$$\partial^\nu f(\bar{x}) := N(L_f(\bar{x}), \bar{x}).$$

3 Optimality conditions for set-constrained problems

Consider the minimization problem (P) with assumption that C is convex.

Theorem 1. *Let $\bar{x}$ be an efficient solution to* P *and an extreme point of C. Assume that $L_f(\bar{x})$ is convex, C is not reduced to $\{\bar{x}\}$ and either one of the following conditions holds*

(i) $\mathrm{int}L_f^<(\bar{x}) \neq \emptyset$ *or f is u.s.c. at a point of $L_f^<(\bar{x})$;*

(ii) $\mathrm{int}C \neq \emptyset$;

(iii) *X is finite dimensional.*

Then

$$\partial^\nu f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}\,. \tag{1}$$

Proof. Since $\bar{x}$ is an extreme point of C and $C \neq \{\bar{x}\}$, the set $C \setminus \{\bar{x}\}$ is convex and nonempty. As $\bar{x}$ is a strict solution to (P), $C \setminus \{\bar{x}\}$ and $L_f(\bar{x})$ are disjoint. For (i), by the Hahn-Banach separation theorem, there exists some $c \in \mathbb{R}$ and $0 \neq u^* \in X^*$ such that the following inequalities hold, for all $w \in L_f(\bar{x})$ and $x \in C \setminus \{\bar{x}\}$,

$$\langle u^*, x - \bar{x}\rangle \geq c \geq \langle u^*, w - \bar{x}\rangle.$$

Since x can be arbitrarily close to $\bar{x}$, we have $c \leq 0$. On the other hand, since we can take $w = \bar{x}$, $c \geq 0$ and hence $c = 0$. Therefore, the left inequality means $u^* \in -N(C, \bar{x})$ and the right one means $u^* \in \partial^\nu f(\bar{x})$, and the result follows.

For (ii) and (iii), we can also apply the separation theorem to get the above inequalities for all $w \in L_f(\bar{x})$ and $x \in C \setminus \{\bar{x}\}$. Similar arguments complete the proof. $\square$

Passing to sufficient conditions, we need the following strict convexity, used in [21] among others. Let X be a normed space. $A \subseteq X$ is called strictly convex at $\bar{x}$ if $\langle x^*, x - \bar{x}\rangle < 0$ for every $x \in A \setminus \{\bar{x}\}$ and $x^* \in N(A, \bar{x}) \setminus \{0\}$. Suggested by a referee, we discuss relations between this strict convexity with some close known notions. Denote $A(x) := cone(A - x)$. $\bar{x} \in A$ is called an extreme point of A if $A(\bar{x}) \cap (-A(\bar{x})) = \{0\}$, and a strictly extreme point of A if $\mathrm{cl}A(\bar{x}) \cap (-\mathrm{cl}A(\bar{x})) = \{0\}$. $\bar{x} \in A$ is said to be an exposed point of A if there exists $x^* \in X^*$ such that $< x^*, x >< < x^*, \bar{x} >$ for all $x \in A \setminus \{\bar{x}\}$, and a strictly exposed point of A if this strict inequality holds for all $x \in \mathrm{cl}A \setminus \{\bar{x}\}$. Moreover, $\bar{x} \in A$ is called strongly strictly exposed point of A provided that x^* satisfies additionally that if $< x^*, x_n > \to < < x^*, \bar{x} >$ for $x_n \in \mathrm{cl}A$ then $x_n \to x$. In [31] the following relations were established.

(i) Strong strict exposedness $\Rightarrow$ strict exposedness $\Rightarrow$ exposedness $\Rightarrow$ extremeness.

(ii) A strictly exposed point is always a strictly extreme point. The converse is true if X is separable and A is convex. A strictly extreme point is incomparable with an exposed point. Example 6 of [31] gave a strictly extreme point which is not an exposed point. Conversely, boundary points of a disk are exposed but not strictly extreme.

(iii) If X is finite dimensional and A is convex, the three notions of strong strict exposedness, strict exposedness and strict extremeness are equivalent.

Now we compare with strict convexity. It is clear that A is strictly convex at $\bar{x}$, then this point is an extreme point. Unfortunately, strict convexity is incomparable with other above-mentioned properties. We give illustrative examples.

(i) The set $A = \{(x, y) : y > 0\} \cup \{(0, 0)\}$ in $\mathbb{R}^2$ is strictly convex at $(0, 0)$, but this point is not strictly extreme. While $(0, 0)$ is a strictly extreme point of $\mathbb{R}^2_+$ but this set is not strictly convex at $(0, 0)$.

(ii) The set A in (i) is strictly convex at $(0, 0)$, but this point is not strictly exposed. On the other hand, $(0, 0)$ is strongly strictly exposed point of $\mathbb{R}^2_+$ but this set is not strictly convex at $(0, 0)$.

We have the following simple sufficient condition.

Theorem 2. *A feasible point $\bar{x}$ is an efficient solution of (P) if (1) is satisfied and either of the following conditions holds*

(i) *either C is strictly convex at $\bar{x}$ or $C \setminus \{\bar{x}\}$ is open;*

(ii) *$L_f(\bar{x}) \setminus \{\bar{x}\}$ is open.*

Proof. Suppose, ad absurdum, $(L_f(\bar{x}) \cap C) \setminus \{\bar{x}\} \neq \emptyset$. The relation (1) implies that there exists $0 \neq u^* \in X^*$ such that

$$\langle u^*, x - \bar{x}\rangle \geq 0 \geq \langle u^*, w - \bar{x}\rangle, \qquad \forall w \in L_f(\bar{x}),\ x \in C,$$

then $\langle u^*, v - \bar{x}\rangle = 0$ for any $v \in (L_f(\bar{x}) \cap C) \setminus \{\bar{x}\}$. For (i), observe first that $C \setminus \{\bar{x}\}$ is open implies that C is strictly convex at $\bar{x}$. Indeed, if $C \setminus \{\bar{x}\}$ is open, it is equal to int$C$. Hence $\langle x^*, x - \bar{x}\rangle < 0$ for every $x \in C \setminus \{\bar{x}\}$ and $x^* \in N(C, \bar{x}) \setminus \{0\}$, i.e. C is strictly convex at $\bar{x}$. Now we have to consider only the case where C is strictly convex at $\bar{x}$. Then, by the strict convexity, $\langle -u^*, x - \bar{x}\rangle < 0$ for all $x \in C \setminus \{\bar{x}\}$, a contradiction. For (ii), let $h \in X$ be nonzero and arbitrary. As $L_f(\bar{x}) \setminus \{\bar{x}\}$ is open, there exists $t > 0$ small enough such that $v + th \in L_f(\bar{x})$. Then

$$t\langle u^*, h\rangle = \langle u^*, v - \bar{x} + th\rangle - \langle u^*, v - \bar{x}\rangle \leq 0.$$

Hence $u^* = 0$, again a contradiction. □

4 Optimality conditions for the mathematical programming problem

Let us consider now the case in which the constraint set C is defined by a finite family of inequalities, so that problem (P) turns into the mathematical programming problem

$$\text{minimize } f(x) \quad \text{subject to } \; g_1(x) \le 0, ..., g_n(x) \le 0. \tag{MP}$$

We denote $g = \max_{1 \le i \le n} g_i$, $C = g^{-1}(-\infty, 0]$, $I = \{i : g_i(\bar{x}) = 0\}$ and $h = \max_{i \in I} g_i$.

Theorem 3. *Assume for problem* (MP) *that*

(i) $L_f^<(\bar{x}) \cup \{\bar{x}\}$ *and* $g_i^{-1}(-\infty, 0]$ *are convex for i=1,...,n;*

(ii) g_i *are u.s.c. at* $\bar{x}$ *for* $i = 1, ..., n$;

(iii) *either of the following regularity conditions holds*

(a) there exists $k \in I$ *such that* $L_{g_k}(\bar{x}) \cap \{x \in X : x \in L_{g_i}^<(\bar{x}), \forall i \in I \setminus \{k\}\} \neq \emptyset$ *(Slater condition);*

(b) X *is complete,* $L_{g_i}(\bar{x})$ *is closed for each* $i \in I$ *and* $\mathbb{R}_+(\Delta - \prod_{i \in I} L_{g_i}(\bar{x}))$ *is a closed subspace, where* $\Delta = \{(x_i)_{i \in I} : \forall j, k \in I; x_j = x_k\}$ *is the diagonal of* X^I.

(iv) *either* X *is finite dimensional or* f *is u.s.c. at some point of* $L_f^<(\bar{x})$.

If $\bar{x}$ *is an efficient solution, which is an extreme point but not a single point of the feasible set, then*

$$\partial^\nu f(\bar{x}) \cap \left(-\sum_{i \in I} \partial^\nu g_i(\bar{x}) \right) \neq \{0\}. \tag{2}$$

This implies the following usual form, for some $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$, *not all zero, such that*

$$0 \in \partial^\nu f(\bar{x}) + \sum_{j=0}^{n} \lambda_j \partial^\nu g_j(\bar{x}), \tag{3}$$

$$\lambda_j g_j(\bar{x}) = 0, j = 1, ..., n. \tag{4}$$

Proof. Observe that C is convex and contained in $L_h(\bar{x})$. So $N(L_h(\bar{x}), \bar{x}) \subseteq N(C, \bar{x})$. To prove the reverse inclusion we show that $T(L_h(\bar{x}), \bar{x}) \subseteq T(C, \bar{x})$. By the assumed convexity we have $T(L_h(\bar{x}), \bar{x}) = \text{clcone}(L_h(\bar{x}) - \bar{x})$, i.e. any $v \in T(L_h(\bar{x}), \bar{x})$ is of the form $\lim t_k(x_k - \bar{x})$, where $t_k > 0$ and $x_k \in L_h(\bar{x})$. On the other hand, let $x \in L_h(\bar{x})$ be arbitrary. If $i \in I$ and $x_t := \bar{x} + t(x - \bar{x})$, then $g_i(x_t) \le 0$ for $t \in [0, 1]$ by the convexity. For $i \notin I$, $\bar{x} \in \text{int } g_i^{-1}(-\infty, 0]$ by the assumed upper semicontinuity. So, for $t > 0$ small enough, $g_i(x_t) \le 0$. Hence $x_t \in C$. Therefore, $t(x - \bar{x}) \in \text{cone}(C - \bar{x})$ for any $x \in L_h(\bar{x})$ and any $t > 0$. It follows that the above-mentioned lim $t_k(x_k - \bar{x})$ belongs to clcone $(C - \bar{x}) = T(C, \bar{x})$. Thus, $T(L_h(\bar{x}), \bar{x}) \subseteq T(C, \bar{x})$, and then $N(C, \bar{x}) \subseteq N(L_h(\bar{x}), \bar{x})$. Thus we have equality.

In case (a) with the Slater condition, one has $L_{g_k}(\bar{x}) \cap (\cap_{i \in I \setminus \{k\}} \mathrm{int} L_{g_i}(\bar{x})) \neq \emptyset$ (by the assumed upper semicontinuity) and hence, by the Moreau-Rockafellar theorem,

$$N(C, \bar{x}) = N(L_h(\bar{x}), \bar{x}) = \sum_{i \in I} N(L_{g_i}(\bar{x}), \bar{x}).$$

In case (b), by Theorem 4.3 of [25], we also have this relation. By (iv) we can apply Theorem 1 to get (2). Taking $\lambda_i \in \mathbb{R}_+ \setminus \{0\}$ arbitrarily for $i \in I$ and $\lambda_i = 0$ for $i \notin I$ we obtain (3) and (4). □

Theorem 4. *Let f be u.s.c. and $\bar{x}$ be a feasible solution of problem* (MP). *Then, relation* (2) *implies that $\bar{x}$ is an efficient solution.*

Proof. Let $D = h^{-1}(-\infty, 0]$. Then, $C \subseteq D$. Observe that $\bar{x}$ is a solution to problem (MP) if and only if $L_f^{<}(\bar{x}) \cap C = \emptyset$. We shall prove a stronger conclusion that $L_f^{<}(\bar{x}) \cap D = \emptyset$. Since f is u.s.c., $L_f^{<}(\bar{x})$ is open. Note that we always have

$$N(D, \bar{x}) \supseteq \sum_{i \in I} N(L_{g_i}(\bar{x}), \bar{x}) = \sum_{i \in I} \partial^{\nu} g_i(\bar{x}).$$

Now applying Theorem 2, we see that $\bar{x}$ is a solution to the following set-constrained problem

$$\text{minimize } f(x) \text{ subject to } x \in D.$$

This completes the proof. □

References

[1] K. J. Arrow and A. C. Enthoven, *Quasi-concave programming*, Econometrica, 29 (1961), pp. 779-800.

[2] H. Attouch and H. Brézis, *Duality for the sum of convex functions in general Banach spaces*, in Aspects of Mathematics and Its Applications, J. A. Barroso, ed., North-Holland, Amsterdam, 1986, pp. 125-133.

[3] J. M. Borwein and Q. J. Zhu, *A survey of subdifferential calculus with applications*, Nonlinear Anal., 38 (1999), pp. 687-773.

[4] A. Daniilidis, N. Hadjisavvas, and J.-E. Martínez-Legaz, *An appropriate subdifferential for quasiconvex functions*, SIAM J. Optim., 12 (2001), pp. 407-420.

[5] J. Dutta, V. Vetrivel, and S. Nanda, *Semi-invex functions and their subdifferentials*, Bull. Austral. Math. Soc., 56 (1997), pp. 385-393.

[6] G. Giorgi and A. Guerraggio, *First order generalized optimality conditions for programming problems with a set constraint*, in Generalized Convexity. Proceedings of the 5th International Workshop on Generalized Convexity, Pécs, Hungary, 1992, S. Komlósi et al., eds., Lecture Notes Econ. Math. Syst. 405, Springer-Verlag, Berlin, 1994, pp. 171-185.

[7] G. Giorgi and A. Guerraggio, *Various types of nonsmooth invex functions*, J. Inform. Optim. Sci., 17 (1996), pp. 137-150.

[8] B. M. Glover and V. Jeyakumar, *Abstract nonsmooth nonconvex programming*, in Proceedings of the 5th International Workshop on Generalized Convexity Held at Janus Pannonius University, Pécs, Hungary, 1992, S. Komlósi et al., eds., Lecture Notes Econ. Math. Syst. 405, Springer-Verlag, Berlin, 1994, pp. 186-210.

[9] H. P. Greenberg and W. P. Pierskalla, *Quasiconjugate function and surrogate duality*, Cahiers du Centre dEtude de Recherche Opérationnelle, 15 (1973), pp. 437-448.

[10] J. M. Gutiérrez Díez, *Infragradients and directions of decrease*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid, 78 (1984), pp. 523-532.

[11] J. Gwinner and V. Jeyakumar, *A solvability theorem and minimax fractional programming*, Z. Oper. Res., 37 (1993), pp. 1-12.

[12] V. Jeyakumar and B. M. Glover, *A new version of Farkas' lemma and global convex maximization*, Appl. Math. Lett., 6 (1993), pp. 39-43.

[13] V. Jeyakumar, W. Oettli, and M. Natividad, *A solvability theorem for a class of quasiconvex mappings with applications to optimization*, J. Math. Anal. Appl., 179 (1993), pp. 537-546.

[14] Z. A. Khan, *On nondifferentiable quasiconvex programming problem*, J. Inform. Optim. Sci., 12 (1991), pp. 57-64.

[15] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[16] J. E. Martínez-Legaz, *Quasiconvex duality theory by generalized conjugation methods*, Optimization, 19 (1988), pp. 603-652.

[17] J.-E. Martínez-Legaz, *On lower subdifferentiable functions*, in Trends in Mathematical Optimization, K. H. Hoffmann et al., eds., Internat. Series Numer. Math. 84, Birkhauser, Basel, 1988, pp. 197-232.

[18] J.-E. Martínez-Legaz, *Weak lower subdifferentials and applications*, Optimization, 21 (1990), pp. 321-341.

[19] J.-E. Martínez-Legaz and S. Romano-Rodríguez, *α-lower subdifferentiable functions*, SIAM J. Optim., 3 (1993), pp. 800-825.

[20] J.-E. Martínez-Legaz and S. Romano-Rodríguez, *Lower subdifferentiability of quadratic functions*, Math. Programming, 60 (1993), pp. 93-113.

[21] J. E. Martínez-Legaz and P. H. Sach, *A new subdifferential in quasiconvex analysis*, J. Convex Anal., 6 (1999), pp. 1-11.

[22] N. T. H. Linh and J.-P. Penot, *Optimality conditions for quasiconvex programs*, SIAM J. Optim. 17 (2006), no. 2, pp. 500-510.

[23] J.-P. Penot, *Are generalized derivatives useful for generalized convex functions?*, in Generalized Convexity, Generalized Monotonicity: Recent Results, J.-P. Crouzeix, J. E. Martinez- Legaz, and M. Volle, eds., Kluwer, Dordrecht, The Netherlands, 1998, pp. 3-59.

[24] J.-P. Penot, *What is quasiconvex analysis?*, Optimization, 47 (2000), pp. 35-110.

[25] J.-P. Penot, *A variational subdifferential for quasiconvex functions*, J. Optim. Theory Appl., 111 (2001), pp. 165-171.

[26] J.-P. Penot, *Characterization of solution sets of quasiconvex programs*, J. Optim. Theory Appl., 117 (2003), pp. 627-636.

[27] J.-P. Penot, *A Lagrangian approach to quasiconvex analysis*, J. Optim. Theory Appl., 117 (2003), pp. 637-647.

[28] J.-P. Penot and P. H. Sach, *Generalized monotonicity of subdifferentials and generalized convexity*, J. Optim. Theory Appl., 94 (1997), pp. 251-262.

[29] J.-P. Penot and M. Volle, *Another duality scheme for quasiconvex problems*, in Trends in Mathematical Optimization, K. H. Hoffmann et al., eds., Internat. Series Numer. Math. 84, Birkhäuser, Basel, Switzerland, 1988, pp. 259-275.

[30] J.-P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res., 15 (1990), pp. 597- 625.

[31] J.-P. Penot and M. Volle, *Surrogate programming and multipliers in quasiconvex programming*, SIAM J. Control Optim., 42 (2003), pp. 1994-2003.

[32] J.-P. Penot and C. Zalinescu, Harmonic sums and duality, J. Convex Anal., 7 (2000), pp. 95-113.

[33] J.-P. Penot and C. Zalinescu, *Elements of quasiconvex subdifferential calculus*, J. Convex Anal., 7 (2000), pp. 243-269.

[34] R. Pini, *Invexity and generalized convexity, Optimization*, 22 (1991), pp. 513-525.

[35] F. Plastria, *Lower subdifferentiable functions and their minimization by cutting planes*, J. Optim. Theory Appl., 46 (1985), pp. 37-53.

[36] P. Q. Khanh, H. T. Quyen, J. C. Yao, *Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials*, European J. Operations Research, 212 (2011), pp. 235241